Splitting an Argumentation Framework

Ringo Baumann

Universität Leipzig, Johannisgasse 26, 04103 Leipzig, Germany,
baumann@informatik.uni-leipzig.de

Abstract. Splitting results in non-mononotonic formalisms have a long tradition. On the one hand, these results can be used to improve existing computational procedures, and on the other hand they yield deeper theoretical insights into how a non-monotonic approach works. In the 90's Lifschitz and Turner [1, 2] proved splitting results for logic programs and default theory. In this paper we establish similar results for Dung style argumentation frameworks (AFs) under the most important semantics, namely stable, preferred, complete and grounded semantics. Furthermore we show how to use these results in dynamical argumentation.

1 Introduction

Argumentation frameworks (AFs) as introduced in the seminal paper of Dung [3] are static. Since argumentation is a dynamic process, it is natural to investigate the dynamic behavior of AFs. In recent years the first publications appeared that deal with the problem of dynamical argumentation. One main direction in this field of research is to study the problem of how extensions of an AF may change if new (old) arguments and/or attack relations are added (deleted) (see, e.g. [4] and the references therein). A further question in this context is how to construct the extensions of an expanded AF by using the (already computed) acceptable sets of arguments of the initial AF. A solution to this problem is obviously of high interest from a computational point of view, especially in case of a huge number of arguments and attacks between them.

In 1994 Lifschitz and Turner [1] published splitting results for logic programs. They have shown that, under certain conditions, a logic program P can be split into two parts P_1 and P_2 such that the computation of an answer set can be considerably simplified: one computes an answer set E_1 of P_1, uses E_1 to modify P_2, computes an answer set E_2 of the modification of P_2 and then simply combines E_1 and E_2. We conveyed this idea to Dung's argumentation frameworks. It turns out that for stable semantics the result is similar to logic programs. However, for preferred, complete and grounded semantics, a more sophisticated modification is needed which takes into account that arguments may be neither accepted nor refuted in extensions.

Our definition of a splitting is closely connected with a special class of expansions, so-called *weak expansions*. This interrelation allows us to transfer our splitting results into the field of dynamical argumentation.

The paper is organized as follows: Section 2 reviews the necessary definitions at work in argumentation frameworks. The third section introduces new concepts like: Splitting, Expansion, Reduct, Undefined Set and Modification. Section 4, the main part of this paper, contains the splitting results for stable, preferred, complete and grounded semantics. Furthermore we compare the splitting theorem with a former monotonicity result. In section 5 we turn to dynamical argumentation. We concentrated on two issues: How to reuse already computed extensions and new terms of equivalence between two AFs. Finally, in section 6 we discuss related results and present our conclusions.

2 Preliminaries

We start with a brief review of the relevant definitions in argumentation theory.

Definition 1. *An argumentation framework $\mathcal{A}$ is a pair (A, R), where A is a non-empty finite set whose elements are called arguments and $R \subseteq A \times A$ a binary relation, called the attack relation.*

If $(a, b) \in R$ holds we say that a attacks b, or b is attacked by a. In the following we consider a fixed countable set $\mathcal{U}$ of arguments, called the *universe*. Quantified formulae refer to this universe and all denoted sets are finite subsets of $\mathcal{U}$ or $\mathcal{U} \times \mathcal{U}$ respectively. We introduce the union for two AFs $\mathcal{F} = (A_F, R_F)$ and $\mathcal{G} = (A_G, R_G)$ as expected, namely $\mathcal{F} \cup \mathcal{G} = (A_F \cup A_G, R_F \cup R_G)$. Furthermore we will use the following abbreviations.

Definition 2. *Let $\mathcal{A} = (A, R)$ be an AF, B and B' subsets of A and $a \in A$. Then*

1. $(B, B') \mathrel{\bar{\in}} R \Leftrightarrow_{def} \exists b \exists b' : b \in B \wedge b' \in B' \wedge (b, b') \in R$,
2. *a is defended by B in $\mathcal{A}$* $\Leftrightarrow_{def} \forall a' : a' \in A \wedge (a', a) \in R \rightarrow (B, \{a'\}) \mathrel{\bar{\in}} R$,
3. *B is conflict-free in $\mathcal{A}$* $\Leftrightarrow_{def} (B, B) \mathrel{\bar{\notin}} R$,
4. $cf(\mathcal{A}) = \{C \,|\, C \subseteq A, C$ *conflict-free in* $\mathcal{A}\}$.

Semantics of argumentation frameworks specify certain conditions for selecting subsets of a given AF $\mathcal{A}$. The selected subsets are called *extensions*. The set of all extensions of $\mathcal{A}$ under semantics $\mathcal{S}$ is denoted by $\mathcal{E}_{\mathcal{S}}(A)$. We consider the classical (stable, preferred, complete, grounded [3]) and the ideal semantics [5].

Definition 3. *Let $\mathcal{A} = (A, R)$ be an AF and $E \subseteq A$. E is a*

1. *stable extension ($E \in \mathcal{E}_{st}(\mathcal{A})$) iff*
   $E \in cf(\mathcal{A})$ and for every $a \in A \backslash E$, $(E, \{a\}) \mathrel{\bar{\in}} R$ holds,
2. *admissible extension*[1] *($E \in \mathcal{E}_{ad}(\mathcal{A})$) iff*
   $E \in cf(\mathcal{A})$ and each $a \in E$ is defended by E in $\mathcal{A}$,
3. *preferred extension (i.e. $E \in \mathcal{E}_{pr}(\mathcal{A})$) iff*
   $E \in \mathcal{E}_{ad}(\mathcal{A})$ and for each $E' \in \mathcal{E}_{ad}(\mathcal{A})$, $E \not\subset E'$ holds,

[1] Note that it is more common to speak about admissible sets instead of the admissible semantics. For reasons of unified notation we used the uncommon version.

4. *complete extension* ($E \in \mathcal{E}_{co}(\mathcal{A})$) *iff* $E \in \mathcal{E}_{ad}(\mathcal{A})$ *and for each* $a \in A$ *defended by* E *in* $\mathcal{A}$, $a \in E$ *holds,*
5. *grounded extension* ($E \in \mathcal{E}_{gr}(\mathcal{A})$) *iff* $E \in \mathcal{E}_{co}(\mathcal{A})$ *and for each* $E' \in \mathcal{E}_{co}(\mathcal{A})$, $E' \not\subset E$ *holds,*
6. *ideal extension of* $\mathcal{A}$ ($E \in \mathcal{E}_{id}(\mathcal{A})$) *iff* $E \in \mathcal{E}_{ad}(\mathcal{A})$, $E \subseteq \bigcap_{P \in \mathcal{E}_{pr}(\mathcal{A})} P$ *and for each* $A \in \mathcal{E}_{ad}(\mathcal{A})$ *w.t.p.* $A \subseteq \bigcap_{P \in \mathcal{E}_{pr}(\mathcal{A})} P$ *holds* $E \not\subset A$.

3 Formal Foundation

In this section we will develop the technical tools which are needed to prove the splitting results.

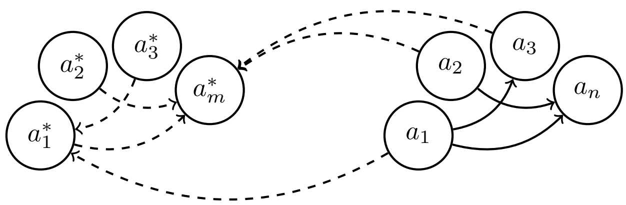
3.1 Splitting and Expansion

Definition 4. *Let* $\mathcal{A}_1 = (A_1, R_1)$ *and* $\mathcal{A}_2 = (A_2, R_2)$ *be AFs such that* $A_1 \cap A_2 = \emptyset$. *Let* $R_3 \subseteq A_1 \times A_2$. *We call the tuple* $(\mathcal{A}_1, \mathcal{A}_2, R_3)$ *a splitting of the argumentation framework* $\mathcal{A} = (A_1 \cup A_2, R_1 \cup R_2 \cup R_3)$.

For short, a splitting of a given AF $\mathcal{A}$ is a partition in two disjoint AFs $\mathcal{A}_1$ and $\mathcal{A}_2$ such that the remaining attacks between $\mathcal{A}_1$ and $\mathcal{A}_2$ are restricted to a single direction. In [6] we studied the dynamical behavior of extensions of an AF. Therefore we introduced a special class of expansions of AFs, so-called *normal expansions*. *Weak* and *strong expansions* are two different subclasses of these expansions. After a short review of the definitions we will show that these kinds of expansions and the introduced splitting definition are in a sense two sides of the same coin. This observation allows us to convey splitting results into dynamical argumentation and vice versa.

Definition 5. *An AF* $\mathcal{A}^*$ *is an expansion of AF* $\mathcal{A} = (A, R)$ *iff* $\mathcal{A}^*$ *can be represented as* $(A \cup A^*, R \cup R^*)$ *for some nonempty* A^* *disjoint from* A *and some (possibly empty)* R^* *disjoint from* R . *Such an expansion is called to be*

1. *normal* ($\mathcal{A} \prec^N \mathcal{A}^*$) *iff* $\forall ab \ \ ((a, b) \in R^* \rightarrow a \in A^* \lor b \in A^*)$,
2. *strong* ($\mathcal{A} \prec_S^N \mathcal{A}^*$) *iff* $\mathcal{A} \prec^N \mathcal{A}^*$ *and* $\forall ab \ \ ((a, b) \in R^* \rightarrow \neg(a \in A \land b \in A^*))$,
3. *weak* ($\mathcal{A} \prec_W^N \mathcal{A}^*$) *iff* $\mathcal{A} \prec^N \mathcal{A}^*$ *and* $\forall ab \ \ ((a, b) \in R^* \rightarrow \neg(a \in A^* \land b \in A))$.



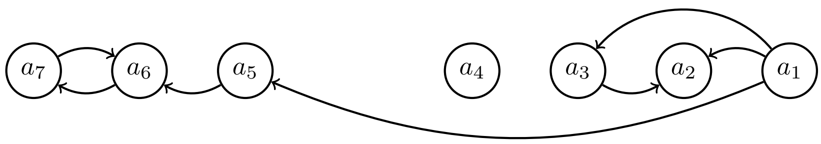
The figure above illustrates a weak expansion[2]. The dashed arrows represent the additional attack relation R^*. The following proposition establishes the connection between splittings and weak expansions. Note that this property is pretty obvious. Being aware of this fact, we still present it in the form of a proposition.

Proposition 1. *If $(\mathcal{A}_1, \mathcal{A}_2, R_3)$ is a splitting of $\mathcal{A}$, then $\mathcal{A}$ is a weak expansion of $\mathcal{A}_1$. Vice versa, if $\mathcal{A} = (A, R)$ is a weak expansion of $\mathcal{A}_1 = (A_1, R_1)$, then $(\mathcal{A}_1, \mathcal{A}_2, R_3)$ with $\mathcal{A}_2 = (A \backslash A_1, R \cap (A \backslash A_1 \times A \backslash A_1))$ and $R_3 = R \cap (A_1 \times A \backslash A_1)$ is a splitting of $\mathcal{A}$.*

3.2 Reduct, Undefined Set and Modification

Now we turn to the central definitions of our paper. The main goal is to establish a connection between the extensions of an AF $\mathcal{A}$ and a given splitting of it. Consider therefore the following example.

Example 1. Let $(\mathcal{A}_1, \mathcal{A}_2, \{(a_1, a_5)\})$ be a splitting of the AF $\mathcal{A}$ below, where $\mathcal{A}_1 = (\{a_1, a_2, a_3, a_4\}, \{(a_1, a_2), (a_1, a_3), (a_3, a_2)\})$ and
$\mathcal{A}_2 = (\{a_5, a_6, a_7\}, \{(a_5, a_6), (a_6, a_7), (a_7, a_6)\})$.



There are two stable extensions of $\mathcal{A}$, namely $E_1 = \{a_1, a_4, a_6\}$ and $E_2 = \{a_1, a_4, a_7\}$. Furthermore we observe that $E' = \{a_1, a_4\}$ and $E'' = \{a_5, a_7\}$ are the unique stable extensions of $\mathcal{A}_1$ and $\mathcal{A}_2$, respectively. Note that we cannot *reconstruct* the extensions E_1 and E_2 out of the extensions E' and E''. This is not very surprising because we do not take into account the attack (a_1, a_5). If we delete the argument a_5 in $\mathcal{A}_2$ which is attacked by E' and then compute the stable extensions in the reduced AF $\mathcal{A}_{2,red} = (\{a_6, a_7\}, \{(a_6, a_7), (a_7, a_6)\})$ we get the "missing" singletons $\{a_6\}$ and $\{a_7\}$. That means E_1 and E_2 are unions of the extensions of $\mathcal{A}_1$ and a reduced version of $\mathcal{A}_2$.

We will see that this observation holds in general for the stable semantics. The following definition of a reduct captures the intuitive idea.

Definition 6. *Let $\mathcal{A} = (A, R)$ be an AF, A' a set disjoint from A, $S \subseteq A'$ and $L \subseteq A' \times A$. The (S, L)-reduct of $\mathcal{A}$, denoted $\mathcal{A}^{S,L}$ is the AF*

$$\mathcal{A}^{S,L} = (A^{S,L}, R^{S,L})$$

where

$$A^{S,L} = \{a \in A \mid (S, \{a\}) \not\in L)\}$$

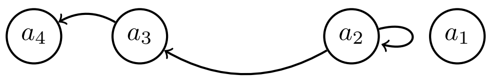
[2] The term is inspired by the fact that added arguments never attack previous arguments (*weak* arguments).

and

$$R^{S,L} = \{(a, b) \in R \mid a, b \in A^{S,L}\}.$$

The intuitionally described reduced version of the AF $\mathcal{A}_2$ in example 1 can be formalized exactly in the following way: $\mathcal{A}_{2,red} = (\{a_6, a_7\}, \{(a_6, a_7), (a_7, a_6)\}) = \mathcal{A}_2^{E',\{(a_1,a_5)\}}$. Unfortunately it turns out that the reduct used above does not obtain the desired properties for other semantics we are interested in. Here is a counterexample:

Example 2. Consider the AF $\mathcal{A} = (\{a_1, a_2, a_3, a_4\}, \{(a_2, a_2), (a_2, a_3), (a_3, a_4)\})$. $\mathcal{A}$ has a splitting $(\mathcal{A}_1, \mathcal{A}_2, \{(a_2, a_3)\})$ with $\mathcal{A}_1 = (\{a_1, a_2\}, \{(a_2, a_2)\})$ and $\mathcal{A}_2 = (\{a_3, a_4\}, \{(a_3, a_4)\})$.



$E_1 = \{a_1\}$ is the unique preferred, complete and grounded extension of $\mathcal{A}$. The same holds for the AF $\mathcal{A}_1$, i.e. $E' = \{a_1\}$. Consider now the $(E', \{(a_2, a_3)\})$-reduct of $\mathcal{A}_2$, that is $\mathcal{A}_2^{E',\{(a_2,a_3)\}} = \mathcal{A}_2$. The reduct establishes the unique extension $E'' = \{a_3\}$ for all considered semantics. Yet the union of E' and E'' differs from E_1.

The problem stems from the fact that the distinction between those arguments which are not in the extension *because they are refuted* (attacked by an accepted argument) and those not in the extension *without being refuted* is not taken care of. The former have no influence on $\mathcal{A}_2$. However, the latter - which we will call undefined in contrast to the refuted ones - indeed have an influence on $\mathcal{A}_2$, as illustrated in the example: the fact that a_2 is undefined in E' leads to the undefinedness of both a_3 and a_4, and this is not captured by the reduct. To overcome this problem, we introduce a simple modification. We can enforce undefinedness of a_3 (and thus of a_4) in $\mathcal{A}_2$ by introducing a self-attack for a_3. More generally, whenever there is an undefined argument a in the extension of the first AF which attacks an argument b in the second AF, we modify the latter so that b is both origin and goal of the attack.

Definition 7. *Let $\mathcal{A} = (A, R)$ be an AF, E an extension of $\mathcal{A}$. The set of arguments undefined with respect to E is*

$$U_E = \{a \in A \mid a \notin E, (E, \{a\}) \not\subseteq R\}.$$

Definition 8. *Let $\mathcal{A} = (A, R)$ be an AF, A' a set disjoint from A, $S \subseteq A'$ and $L \subseteq A' \times A$. The (S, L)-modification of $\mathcal{A}$, denoted $mod_{S,L}(\mathcal{A})$, is the AF*

$$mod_{S,L}(\mathcal{A}) = (A, R \cup \{(b, b) \mid \exists a : a \in S, (a, b) \in L\}).$$

Given a splitting $(\mathcal{A}_1, \mathcal{A}_2, R_3)$ of $\mathcal{A}$, an extension E of $\mathcal{A}_1$ which leaves the set of arguments U_E undefined, we will use $mod_{U_E,R_3}(\mathcal{A}_2^{E,R_3})$ to compute what is missing from E. In case of example 2 we compute the extensions of the

$(\{a_2\}, \{a_2, a_3\})$-modification of the $(E', \{(a_2, a_3)\})$-reduct of $\mathcal{A}_2$, i.e.

$$mod_{\{a_2\},\{a_2,a_3\}}\left(\mathcal{A}_2^{E',\{(a_2,a_3\}}\right) = (\{a_3, a_4\}, \{(a_3, a_3), (a_3, a_4)\}),$$

which establishes the empty set as the unique extension under all considered semantics. Hence the union of E' and $\emptyset$ equals the extension of the initial framework $\mathcal{A}$. Note that, although links are added, under all standard measures of the size of a graph (e.g. number of links plus number of vertices) we have $|\mathcal{A}_1| + |mod_{U_E,R_3}(\mathcal{A}_2^{E,R_3})| \leq |\mathcal{A}|$.

4 Splitting Results

Now we are going to present our formal results. Our splitting results show how to get extensions of the whole AF $\mathcal{A}$ with the help of a splitting. Furthermore we prove that our method is complete, i.e. all extensions are constructed this way. At first we will prove some simple properties of the introduced definitions.

Proposition 2. *Given an AF $\mathcal{A} = (A, R)$ which possesses a splitting $(\mathcal{A}_1, \mathcal{A}_2, R_3)$ s.t. $\mathcal{A}_1 = (A_1, R_1)$ and $\mathcal{A}_2 = (A_2, R_2)$, the following hold:*

1. $E_1 \in \mathcal{E}_{st}(\mathcal{A}_1) \Rightarrow mod_{U_{E_1},R_3}(\mathcal{A}_2^{E_1,R_3}) = \mathcal{A}_2^{E_1,R_3}$,

   (neutrality of the modification w.r.t. the stable reduct)
2. $E \in cf(\mathcal{A}) \Rightarrow E \cap A_1 \in cf(\mathcal{A}_1) \wedge E \cap A_2 \in cf(\mathcal{A}_2^{E\cap A_1,R_3})$,

   (preserving conflict-freeness[intersection])
3. $E_1 \in cf(\mathcal{A}_1) \wedge E_2 \in cf(mod_{U_{E_1},R_3}(\mathcal{A}_2^{E_1,R_3})) \Rightarrow E_1 \cup E_2 \in cf(\mathcal{A})$.

   (preserving conflict-freeness[union])

Proof. 1. if $E_1 \in \mathcal{E}_{st}(\mathcal{A}_1)$ then all arguments in $A_1 \backslash E_1$ are attacked, consequently $U_{E_1} = \emptyset$ and $mod_{U_{E_1},R_3}(\mathcal{A}_2^{E_1,R_3}) = \mathcal{A}_2^{E_1,R_3}$ holds;
2. subsets of conflict-free sets w.r.t. $\mathcal{A}$ are conflict-free w.r.t. all restrictions of $\mathcal{A}$; hence, it is sufficient to show that $E \cap A_1 \subseteq A_1$ (obvious) and $E \cap A_2 \subseteq A_2^{E\cap A_1,R_3}$ holds; assuming the existence of an argument a, s.t. $a \in E \cap A_2 \wedge a \notin A_2^{E\cap A_1,R_3} = \{a \in A_2 \,|\, (E \cap A_1, \{a\}) \not\bar{\in} R_3\}$ leads to $(E, E) \bar{\in} R_3$ which contradicts the conflict-freeness of E in $\mathcal{A}$;
3. we have to show that $(E_1 \cup E_2, E_1 \cup E_2) \not\bar{\in} R_1 \cup R_2 \cup R_3$ holds; E_1 is conflict-free w.r.t. $\mathcal{A}_1$, thus $(E_1, E_1) \not\bar{\in} R_1 \cup R_2 \cup R_3$ because R_2 and R_3 do not contain attacks from A_1 to A_1; E_2 is conflict-free w.r.t. $mod_{U_{E_1},R_3}(\mathcal{A}_2^{E_1,R_3})$, hence conflict-free w.r.t. $\mathcal{A}_2$ because i) R_2 restricted to $A_2^{E_1,R_3} \times A_2^{E_1,R_3}$ is equal to $R_2^{E_1,R_3}$ and ii) the attack-relation of the (U_{E_1}, R_3)-modification of $\mathcal{A}_2^{E_1,R_3}$ is a super-set of $R_2^{E_1,R_3}$; thus $(E_2, E_2) \not\bar{\in} R_1 \cup R_2 \cup R_3$ because R_1 and R_3 do not contain attacks from A_2 to A_2; it obviously holds that $(E_2, E_1) \not\bar{\in} R_1 \cup R_2 \cup R_3$ because none of the relations contain attacks from A_2 to A_1; per definition we have $E_2 \subseteq A_2^{E_1,R_3} = \{a \in A_2 \,|\, (E_1, \{a\}) \not\bar{\in} R_3\}$, consequently it is impossible that $(E_1, E_2) \bar{\in} R_3$ holds, therefore $(E_1, E_2) \not\bar{\in} R_1 \cup R_2 \cup R_3$ because R_2 and R_1 do not contain attacks from A_1 to A_2 □

4.1 Monotonicity Result

In [6] we have proven the following monotonicity result. This theorem will be used to simplify parts of the proof of the splitting theorem. Furthermore we will see that the splitting theorem is a generalization of it.

Theorem 1. *Given an AF $\mathcal{A} = (A, R)$ and a semantics $\mathcal{S}$ satisfying directionality*[3]*, then for all weak expansions $\mathcal{A}^*$ of $\mathcal{A}$ the following holds:*

1. $|\mathcal{E}_{\mathcal{S}}(\mathcal{A})| \leq |\mathcal{E}_{\mathcal{S}}(\mathcal{A}^*)|$,
2. $\forall E \in \mathcal{E}_{\mathcal{S}}(\mathcal{A})\ \exists E^* \in \mathcal{E}_{\mathcal{S}}(\mathcal{A}^*) : E \subseteq E^*$ *and*
3. $\forall E^* \in \mathcal{E}_{\mathcal{S}}(\mathcal{A}^*)\ \exists E_i \in \mathcal{E}_{\mathcal{S}}(\mathcal{A})\ \exists A_i^* \subseteq A^* : E^* = E_i \cup A_i^*$

Adding new arguments and their associated interactions may change the outcome of an AF in a nonmonotonic way. Accepted arguments may become unaccepted and vice versa. The theorem above specifies sufficient conditions (weak expansions + directionality principle) for monotonic behaviour w.r.t. justification state of an argument and cardinality of extensions. Remember that the admissible, complete, grounded, ideal and preferred semantics satisfy the directionality principle (compare [8]).

4.2 Splitting Theorem

Given a splitting of an AF $\mathcal{A}$, the general idea is to compute an extension E_1 of $\mathcal{A}_1$, reduce and modify $\mathcal{A}_2$ depending on what extension we got, and then compute an extension E_2 of the modification of the reduct of $\mathcal{A}_2$. The resulting union of E_1 and E_2 is an extension of $\mathcal{A}$. The second part of theorem proves the completeness of this method, i.e. all extensions are constructed this way.

Theorem 2. *($\sigma \in \{st, ad, pr, co, gr\}$) Let $\mathcal{A} = (A, R)$ be an AF which possesses a splitting $(\mathcal{A}_1, \mathcal{A}_2, R_3)$ with $\mathcal{A}_1 = (A_1, R_1)$ and $\mathcal{A}_2 = (A_2, R_2)$.*

1. *If E_1 is an extension of $\mathcal{A}_1$ and E_2 is an extension of the (U_{E_1}, R_3)-modification of $\mathcal{A}_2^{E_1,R_3}$, then $E = E_1 \cup E_2$ is an extension of $\mathcal{A}$.*

$$\Big(E_1 \in \mathcal{E}_\sigma(\mathcal{A}_1) \wedge E_2 \in \mathcal{E}_\sigma(mod_{U_{E_1},R_3}(\mathcal{A}_2^{E_1,R_3})) \Rightarrow E_1 \cup E_2 \in \mathcal{E}_\sigma(\mathcal{A})\Big)$$

2. *If E is an extension of $\mathcal{A}$, then $E_1 = E \cap A_1$ is an extension of $\mathcal{A}_1$ and $E_2 = E \cap A_2$ is an extension of the (U_{E_1}, R_3)-modification of $\mathcal{A}_2^{E_1,R_3}$.*

$$\Big(E \in \mathcal{E}_\sigma(\mathcal{A}) \Rightarrow E \cap A_1 \in \mathcal{E}_\sigma(\mathcal{A}_1) \wedge E \cap A_2 \in \mathcal{E}_\sigma(mod_{U_{E\cap A_1},R_3}(\mathcal{A}_2^{E\cap A_1,R_3}))\Big)$$

Proof. **(stable)**(1.) by prop. 2.3 we got the conflict-freeness of $E_1 \cup E_2$ in $\mathcal{A}$; we now show that $E_1 \cup E_2$ attacks all outer arguments, i.e. for every $a \in (A_1 \cup A_2)\backslash(E_1 \cup E_2)$ holds: $(E_1 \cup E_2, \{a\}) \,\bar{\in}\, R_1 \cup R_2 \cup R_3$; let a be an element of $A_1\backslash(E_1 \cup E_2)$, thus a is attacked by E_1 because $E_1 \in \mathcal{E}_{st}(\mathcal{A}_1)$ holds; let

[3] Intuitively, the directionality principle prescribes that the acceptability of an argument a is determined only by its attackers (compare [7]).

a be an element of $A_2\backslash(E_1 \cup E_2)$; we have to consider two cases because A_2 is the disjoint union of $\{a \in A_2 \,|(E_1, \{a\}) \not\bar{\in} R_3\} \cup \{a \in A_2 \,|(E_1, \{a\}) \bar{\in} R_3\}$; if a is an element of the second set we have nothing to show; let a be an element the first set, namely $\{a \in A_2 \,|(E_1, \{a\}) \not\bar{\in} R_3\} = A_2^{E_1,R_3}$; thus a is attacked by E_2 because $E_2 \in \mathcal{E}_{st}(mod_{U_{E_1},R_3}(\mathcal{A}_2^{E_1,R_3}))$ and $mod_{U_{E_1},R_3}(\mathcal{A}_2^{E_1,R_3}) = \mathcal{A}_2^{E_1,R_3}$ (prop. 2.1) holds;
(2.) at first we will show that $E \cap A_1 = E_1$ is a stable extension of $\mathcal{A}_1$, i.e. $E_1 \in \mathcal{E}_{st}(\mathcal{A}_1)$; conflict-freeness w.r.t. $\mathcal{A}_1$ follows from prop. 2.2; we only have to show that for every $a \in A_1\backslash E_1$ holds: $(E_1, \{a\}) \bar{\in} R_1$; assume not, i.e. $\exists a \in A_1\backslash(E \cap A_1) : (E \cap A_1, \{a\}) \not\bar{\in} R_1$; consequently $(E, \{a\}) \not\bar{\in} R_1 \cup R_2 \cup R_3$ and this contradicts $E \in \mathcal{E}_{st}(\mathcal{A})$;
with prop. 2.1 and 2.2 we get $E \cap A_2 = E_2 \in cf(mod_{U_{E_1},R_3}(\mathcal{A}_2^{E_1,R_3}))$; we now show that $E \cap A_2$ attacks all outer arguments, i.e. for every $a \in A_2^{E_1,R_3}\backslash E_2$ holds: $(E_2, \{a\}) \bar{\in} R_2^{E_1,R_3}$; assuming the contrary, i.e. $\exists a \in A_2^{E_1,R_3}\backslash E_2$: $(E_2, \{a\}) \not\bar{\in} R_2^{E_1,R_3}$ leads directly to $(E, \{a\}) \not\bar{\in} R_1 \cup R_2 \cup R_3$ which contradicts the fact that E is a stable extension of $\mathcal{A}$; $(E, \{a\}) \not\bar{\in} R_3$ because $a \in \{a^* \in A_2 \,|\, (E_1, \{a^*\}) \not\bar{\in} R_3\}$ holds; furthermore $(E, \{a\}) \not\bar{\in} R_2$ because $R_2^{E_1,R_3} \subseteq R_2$ holds and the remaining attacks in R_2 do not contain attacks from $A_2^{E_1,R_3}$ to $A_2^{E_1,R_3}$; the R_1 - case is obvious because $a \in A_2$ holds
(admissible) (1.) admissible sets are conflict-free per definition, hence conflict-freeness of $E_1 \cup E_2$ in $\mathcal{A}$ is given by prop. 2.3;
we have to show that each element of $E_1 \cup E_2$ is defended by $E_1 \cup E_2$ in $\mathcal{A}$, i.e. for each $a \in E_1 \cup E_2$ holds: if $(b, a) \in R_1 \cup R_2 \cup R_3$, then $(E_1 \cup E_2, \{b\}) \bar{\in} R_1 \cup R_2 \cup R_3$; let a be an element of E_1; if a is attacked by an element b, then $b \in A_1$ and $(b, a) \in R_1$ holds, hence the admissibility of E_1 in $\mathcal{A}_1$ guarentees the defence of a by $E_1 \cup E_2$ in $\mathcal{A}$; let a be an element of E_2; we have to consider two cases, namely $b \in A_1$ and $b \in A_2$; assuming $b \in A_1$ yields $(b, a) \in R_3$; we have already shown the conflict-freeness of $E_1 \cup E_2$, hence b has to be an element of $A_1\backslash E_1$; again two cases arise, either $b \in U_{E_1}$ or $b \notin U_{E_1}$; the first case is not possible because elements which are attacked by undefined arguments w.r.t. E_1 get additional self-attacks in the modification, hence these elements can not be in the conflict-free extension E_2; the second case, namely $b \notin U_{E_1}$ (and $b \notin E_1$) can be true but if a is attacked by b, than $(E_1, \{b\}) \bar{\in} R_1$ holds per definition of the undefined arguments w.r.t. E_1; consider now $b \in A_2$ and $(b, a) \in R_2$; we have to distinguish two cases, namely $b \in A_2^{E_1,R_3}$ or $b \notin A_2^{E_1,R_3}$; the counterattack of b by $E_1 \cup E_2$ in the first case is assured because E_2 defends its elements in $mod_{U_{E_1},R_3}(\mathcal{A}_2^{E_1,R_3})$, hence E_2 defends its elements in $\mathcal{A}_2^{E_1,R_3}$ (the deleted self-attacks do not change the defense-state of elements in E_2); that means there is only one case left, namely $(b, a) \in R_2$ and $b \notin A_2^{E_1,R_3}$, i.e. $b \in \{a \in A_2 \,|(E_1, \{a\}) \bar{\in} R_3\}$; hence b is counterattacked by E_1 which completes the proof that a is defended by $E_1 \cup E_2$ in $\mathcal{A}$;
(2.) using that admissible semantics satisfying directionality we conclude immediately $E \cap A_1 = E_1$ is an admissible extension of $\mathcal{A}_1$, i.e. $E_1 \in \mathcal{E}_{ad}(\mathcal{A}_1)$ (compare theorem 1.3);

now we want to show that $E \cap A_2 = E_2 \in \mathcal{E}_{ad}(mod_{U_{E_1},R_3}(\mathcal{A}_2^{E_1,R_3}))$ holds; at first we note that E_2 is indeed a subset of $A_2^{E_1,R_3}$ (compare prop. 2.2); we now show the conflict-freeness of E_2 w.r.t. $mod_{U_{E_1},R_3}(\mathcal{A}_2^{E_1,R_3})$, i.e. $(E_2, E_2) \not\subseteq R_2^{E_1,R_3} \cup \{(b,b) \mid a \in U_{E_1}, (a,b) \in R_3\}$; again prop. 2.2 justifies $(E_2, E_2) \not\subseteq R_2^{E_1,R_3}$; $(E_2, E_2) \not\subseteq \{(b,b) \mid a \in U_{E_1}, (a,b) \in R_3\}$ holds because if there is a b in E_2 which get a self-attack by the modification, than b has to be attacked by an undefined element $a \in U_{E_1}$; but this means that E does not defend its elements in $\mathcal{A}$ because a is per definition unattacked by E_1; at last we want to show that E_2 defends all its elements in $mod_{U_{E_1},R_3}(\mathcal{A}_2^{E_1,R_3})$; assume $a \in E_2 \wedge b \in A_2^{E_1,R_3} = \{b \in A_2 \mid (E_1, \{b\}) \not\subseteq R_3\} \wedge (b,a) \in R_2^{E_1,R_3} \cup \{(b,b) \mid a \in U_{E_1}, (a,b) \in R_3\}$; we observe that $a \neq b$ holds, because i) E is conflict-free and $R_2^{E_1,R_3} \subseteq R_2$ holds and ii) the additional self-attacks of the modification do not involve elements of E_2 because assuming this contradicts again the fact that E defends all its elements in $\mathcal{A}$; thus $(b,a) \in R_2^{E_1,R_3} \subseteq R_2$ holds, consequently there is a $c \in E : (c,b) \in R_1 \cup R_2 \cup R_3$; it holds that $c \notin E \cap A_1$ because of $b \in A_2^{E_1,R_3}$, hence $c \in E \cap A_2 \wedge (c,b) \in R_2$ holds; this implies $(c,b) \in R_2^{E_1,R_3}$ which completes the proof

(preferred) (1.) we have to show that $E_1 \cup E_2 \in \mathcal{E}_{pr}(\mathcal{A})$, i.e. $E_1 \cup E_2$ is admissible (already shown since each preferred extension is admissible) and maximal w.r.t. the set inclusion; assume not, hence there is a $E^* \in \mathcal{E}_{ad}(\mathcal{A}) : E_1 \cup E_2 \subset E^*$; thus at least one of the following two cases is true: $E_1 \subset E^* \cap A_1$ or $E_2 \subset E^* \cap A_2$; assuming the first one contradicts the maximality of E_1 because $E^* \cap A_1$ is an admissible extension of $\mathcal{A}_1$; we observe that $E^* \cap A_1 = E_1$ holds; consider now $E_2 \subset E^* \cap A_2$; using the second part of the splitting theorem for admissible sets yields $E^* \cap A_2 \in \mathcal{E}_{ad}(mod_{U_{E_1},R_3}(\mathcal{A}_2^{E_1,R_3}))$ which contradicts the maximality of E_2; hence, we have proven that $E_1 \cup E_2 \in \mathcal{E}_{pr}(\mathcal{A})$ holds;
(2.) let E be an preferred extension of $\mathcal{A}$; using that the preferred semantics satisfies directionality we conclude $E \cap A_1 \in \mathcal{E}_{pr}(\mathcal{A}_1)$ (theorem 1.3); admissibility of $E \cap A_2$ w.r.t. $mod_{U_{E \cap A_1},R_3}(\mathcal{A}_2^{E \cap A_1,R_3})$ is obvious since every preferred extension is admissible (theorem 2.2 [admissible case]); assume now the existence of an $E_2^* \in \mathcal{E}_{ad}(mod_{U_{E \cap A_1},R_3}(\mathcal{A}_2^{E \cap A_1,R_3})) : E \cap A_2 \subset E_2^*$; thus $(E \cap A_1) \cup E_2^*$ is admissible w.r.t. $\mathcal{A}$ (theorem 2.1 [admissible case]) which contradicts the maximality of E and we are done

(complete) (1.) we have to show that $E_1 \cup E_2 \in \mathcal{E}_{co}(\mathcal{A})$, i.e. $E_1 \cup E_2$ is admissible (already shown since every complete extension is admissible) and for each $a \in A_1 \cup A_2$ which is defended by $E_1 \cup E_2$ in $\mathcal{A}$ holds: $a \in E_1 \cup E_2$; assume not, hence $\exists a \in (A_1 \cup A_2) \backslash (E_1 \cup E_2) : a$ is defended by $E_1 \cup E_2$ in $\mathcal{A}$; assuming that $a \in A_1 \backslash (E_1 \cup E_2)$ holds contradicts $E_1 \in \mathcal{E}_{co}(\mathcal{A}_1)$; so let $a \in A_2 \backslash (E_1 \cup E_2)$ be true; at first we observe that $a \in A_2^{E_1,R_3}$ holds because of the conflict-freeness of E_1 w.r.t. $\mathcal{A}_1$; we have to consider two attack-scenarios: a) a is attacked by arguments in $A_2 \backslash A_2^{E_1,R_3}$ (and obviously defended by E_1 in $\mathcal{A}$); the reduct-relation do not contain such attacks, hence every "attack" is counterattacked by E_2 in $mod_{U_{E_1},R_3}(\mathcal{A}_2^{E_1,R_3})$; b) a is attacked by arguments in $A_2^{E_1,R_3} \backslash E_2$; hence it must be defended by elements of E_2 in $\mathcal{A}_2^{E_1,R_3}$, thus de-

fended by E_2 in $mod_{U_{E_1},R_3}(\mathcal{A}_2^{E_1,R_3})$ because the corresponding attack-relation do not delete such counterattacks; altogether we have shown that $a \in E_2$ holds, hence $E_1 \cup E_2 \in \mathcal{E}_{co}(\mathcal{A})$ is proven;
(2.) assume $E \in \mathcal{E}_{co}(\mathcal{A})$; using that the complete semantics satisfies directionality we conclude $E \cap A_1 \in \mathcal{E}_{co}(\mathcal{A}_1)$ (theorem 1.3); admissibility of $E \cap A_2$ w.r.t. $mod_{U_{E\cap A_1},R_3}(\mathcal{A}_2^{E\cap A_1,R_3}))$ holds since complete extensions are admissible (theorem 2.2 [admissible case]); supposing $\exists a \in A_2^{E\cap A_1,R_3} \backslash E \cap A_2$: a is defended by $E \cap A_2$ in $mod_{U_{E\cap A_1},R_3}(\mathcal{A}_2^{E\cap A_1,R_3})$ contradicts the completeness of E in $\mathcal{A}$ because possible attackers of a are elements of $A_2^{E\cap A_1,R_3}$ which are counterattacked by $E \cap A_2$; these counterattacks are not added by the modification, hence a is defended by $E \cap A_2$ in $\mathcal{A}_2^{E\cap A_1,R_3}$; furthermore a is defended by E in $\mathcal{A}$ (further attackers are counterattacked by $E \cap A_1$) and again we conclude $E \notin \mathcal{E}_{co}(\mathcal{A})$
(grounded) (1.) we have to show that $E_1 \cup E_2 \in \mathcal{E}_{gr}(\mathcal{A})$, i.e. $E_1 \cup E_2$ is a complete extension of $\mathcal{A}$ (already shown since each grounded extensions is complete) and furthermore it is minimal w.r.t. the set inclusion; assume not, hence there is a set $E^* \in \mathcal{E}_{co}(\mathcal{A})$: $E^* \subset E_1 \cup E_2$; we will show that the following two cases are impossible: i) $E^* \cap A_1 \subset E_1$ or ii) $E^* \cap A_2 \subset E_2$; the first case contradict directly the minimality of E_1 w.r.t. $\mathcal{A}_1$; we observe that $E^* \cap A_1 = E_1$ holds, hence $E^* \cap A_2$ is a complete extension of $mod_{U_{E_1},R_3}(\mathcal{A}_2^{E_1,R_3})$ which contradicts the minimality of E_2;
(2.) let E be a grounded extension of $\mathcal{A}$; using that the grounded semantics satisfies the directionality principle we deduce directly $E \cap A_1 \in \mathcal{E}_{gr}(\mathcal{A}_1)$ (theorem 1.3); assume now the existence of $E_2^* \in \mathcal{E}_{co}(mod_{U_{E\cap A_1},R_3}(\mathcal{A}_2^{E\cap A_1,R_3}))$: $E_2^* \subset E \cap A_2$, thus $E_2^* \cup (E \cap A_1)$ is a complete extension of $\mathcal{A}$ and of course a proper subset of E (which contradicts the minimality of E) □

The splitting theorem obviously strengthens the outcome of the monotonicity result for the admissible, preferred, grounded and complete semantics which all satisfy the directionality principle. We do not only know that an old belief set is contained in a new one and furthermore every new belief set is the union of an old one and a (possibly empty) set of new arguments but rather that every new belief set is the union of an old one and an extension of the corresponding modified reduct and vice versa. The cardinality inequality of the monotonicity result (theorem 1.1) can be strengthened in the following way.

Corollary 1. *Let $(\mathcal{A}_1, \mathcal{A}_2, R_3)$ be a splitting of the argumentation framework $\mathcal{A}^* = (A_1 \cup A_2, R_1 \cup R_2 \cup R_3)$ and $\sigma \in \{ad, pr, co, gr\}$. The following inequality holds:*

$$|\mathcal{E}_\sigma(\mathcal{A})| \leq \sum_{E_i \in \mathcal{E}_\sigma(\mathcal{A})} \left| \mathcal{E}_\sigma \left(mod_{U_{E_i},R_3}(\mathcal{A}_2^{E_i,R_3}) \right) \right| = |\mathcal{E}_\sigma(\mathcal{A}^*)| \, .$$

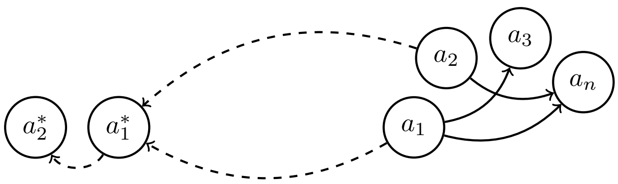
5 Dynamical Argumentation

5.1 Computing Extensions

Since argumentation is a dynamic process, it is natural to investigate dynamic behavior in this context. Obviously the set of extensions of an AF may change if new arguments and their corresponding interactions are added. Computing the justification state of an argument from scratch each time new information is added is very inefficient. Note that in general, new arguments occur as a response, i.e., an attack, to a former argument. In this situation the former extensions are not reusable because in [6] we have shown a possibility result concerning the problem of enforcing of extensions which proves that every conflict-free subset of former arguments may belong to a new extension.

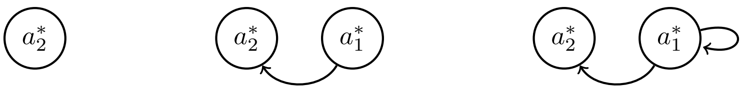
The splitting results allow us to reuse already computed extensions in case of weak expansions. Being aware of the remark above, we emphasize that weak expansions are not only a theoretical situation in argumentation theory. The initial arguments may be arguments which advance higher values[4] than the further arguments. The following dynamical argumentation scenario exemplifies how to use our splitting results.

Example 3. Given an AF $\mathcal{A} = (\{a_1, ..., a_n\}, R)$ and its set of extensions $\mathcal{E}_\sigma(\mathcal{A}) = \{E_1, ..., E_m\}$ ($\sigma \in \{pr, co, gr\}$). Consider now additional *new* arguments a_1^* and a_2^*, where a_1^* is attacked by the *old* arguments a_1 and a_2. Furthermore a_2^* is defeated by a_1^*.



What are the extensions of the expanded AF $\mathcal{A}^* = (A \cup \{a_1^*, a_2^*\}, R \cup \{(a_1, a_1^*), (a_2, a_1^*), (a_1^*, a_2^*)\})$? Since $\mathcal{A}^*$ is a weak expansion of $\mathcal{A}$ we may apply the splitting theorem. Given an extension E_i we construct the $(U_{E_i}, \{(a_1, a_1^*)(a_2, a_1^*)\})$-modification of $(\{a_1^*, a_2^*\}, \{(a_1^*, a_2^*)\})^{E_i, \{(a_1, a_1^*)(a_2, a_1^*)\}}$. The following three cases arise: (1) a_1 or a_2 is an element of E_i, (2) a_1 and a_2 are not in E_i and not in U_{E_i}, and (3) a_1 and a_2 are not in E_i and at least one of them is in U_{E_i}.

The AFs below are the resulting modifications in these three cases. In the first case the argument a_1^* disappears because a_1^* is attacked by an element of the extension E_i (reduct-definition). In the second and third case the arguments a_1^* and a_2^* survive because they are not attacked by E_i. Furthermore in the last case we have to add a selfloop for a_1^* since a_1 or a_2 are undefined (= not attacked) w.r.t. E_i.



The resulting preferred, complete and grounded extensions of the modifications are easily determinable, namely $\{a_2^*\}$ in the first case, $\{a_1^*\}$ in the second

[4] Compare the idea of "attack-succeed" in Value Based Argumentation Frameworks [8].

and the empty set in the last case. Now we can construct extensions of the expanded AF $\mathcal{A}^*$ by using the already computed extensions of $\mathcal{A}$, namely (1) $E_i \cup \{a_2^*\} \in \mathcal{E}_\sigma(\mathcal{A}^*)$, (2) $E_i \cup \{a_1^*\} \in \mathcal{E}_\sigma(\mathcal{A}^*)$, and (3) $E_i \in \mathcal{E}_\sigma(\mathcal{A}^*)$. Due to the completeness of the splitting method we constructed all extensions of $\mathcal{A}^*$.

We want to remark that the splitting results also provide a new possibility to compute extensions in a static AF. This work is still in progress and is not part of this paper.

5.2 Terms of Equivalence

Oikarinen and Woltran [9] extended the notion of equivalence between two AFs (which holds, if they possess the same extensions) to strong equivalence. Strong equivalence between to AFs $\mathcal{F}$ and $\mathcal{G}$ is fullfilled if for all AFs $\mathcal{H}$ holds that $\mathcal{F}$ conjoined with $\mathcal{H}$ and $\mathcal{G}$ conjoined with $\mathcal{H}$ are equivalent. Furthermore they establish criteria to decide strong equivalence. These characterizations are based on syntactical equality of so-called *kernels*.

The following definition weakens the strong equivalence notion w.r.t. weak expanions. We will present a characterization for stable semantics.[5]

Definition 9. *Two AFs $\mathcal{F}$ and $\mathcal{G}$ are weak expansion equivalent to each other w.r.t. a semantics σ, in symbols $\mathcal{F} \equiv^\sigma_{\prec_W^N} \mathcal{G}$, iff for each AF $\mathcal{H}$ s.t.*

- $\mathcal{F} = \mathcal{F} \cup \mathcal{H}$ *or* $\mathcal{F} \prec_W^N \mathcal{F} \cup \mathcal{H}$ *and*
- $\mathcal{G} = \mathcal{G} \cup \mathcal{H}$ *or* $\mathcal{G} \prec_W^N \mathcal{G} \cup \mathcal{H}$,

$\mathcal{E}_\sigma(\mathcal{F} \cup \mathcal{H}) = \mathcal{E}_\sigma(\mathcal{G} \cup \mathcal{H})$ *holds.*

Proposition 3. *For any AFs $\mathcal{F} = (A_F, R_F)$ and $\mathcal{G} = (A_G, R_G)$: $\mathcal{F} \equiv^{st}_{\prec_W^N} \mathcal{G}$ iff*

- $A_F = A_G$ *and* $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G})$ *or*
- $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G}) = \emptyset$.

Proof. ($\Leftarrow$) let $\mathcal{H} = (A_H, R_H)$ be an AF s.t. $\mathcal{F} \prec_W^N$ (*or* $=$) $\mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \prec_W^N$ (*or* $=$) $\mathcal{G} \cup \mathcal{H}$ holds; we show $E \in \mathcal{E}_{st}(\mathcal{F} \cup \mathcal{H})$ implies $E \in \mathcal{E}_{st}(\mathcal{G} \cup \mathcal{H})$; 1^{st} case: assume $A_F = A_G$ and $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G})$; if $\mathcal{F} = \mathcal{F} \cup \mathcal{H}$, then $\mathcal{G} = \mathcal{G} \cup \mathcal{H}$ is implied because $A_F = A_G$ was assumed; that means, $\mathcal{E}_{st}(\mathcal{F} \cup \mathcal{H}) = \mathcal{E}_{st}(\mathcal{G} \cup \mathcal{H})$ holds; from now on we may assume that $\mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \cup \mathcal{H}$ are indeed weak expansions of $\mathcal{F}$ or $\mathcal{G}$; by prop. 1 we get that $(\mathcal{F}, \mathcal{A}_2, R_3)$ and $(\mathcal{G}, \mathcal{A}_2, R_3)$ are splittings of $\mathcal{F} \cup \mathcal{H}$ or $\mathcal{G} \cup \mathcal{H}$, whereas $\mathcal{A}_2 = (A_H \backslash A_F, R_H \cap (A_H \backslash A_F \times A_H \backslash A_F))$ and $R_3 = R_H \cap (A_F \times A_H \backslash A_F)$; given $E \in \mathcal{E}_{st}(\mathcal{F} \cup \mathcal{H})$, theorem 2.2 implies 1. $E \cap A_F \in \mathcal{E}_{st}(\mathcal{F})$ and 2. $E \cap (A_H \backslash A_F) \in \mathcal{E}_{st}\left(mod_{U_{E \cap A_F}, R_3}(\mathcal{A}_2^{E \cap A_F, R_3})\right)$; the first statement implies $E \cap A_G \in \mathcal{E}_{st}(\mathcal{G})$ since $A_F = A_G$ and $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G})$ was asssumed; with prop. 2.1 we derive $E \cap (A_H \backslash A_F) \in \mathcal{E}_{st}\left(\mathcal{A}_2^{E \cap A_F, R_3}\right)$ since $E \cap A_F$ is a stable extension of $\mathcal{F}$; using that A_F equals A_G we get

[5] The remaining semantics are left for future work.

$E \cap (A_H \backslash A_G) \in \mathcal{E}_{st}\left(\mathcal{A}_2^{E\cap A_G, R_3}\right)$; again with prop. 2.1 we follow $E \cap (A_H \backslash A_G) \in \mathcal{E}_{st}\left(mod_{U_{E\cap A_G}, R_3}(\mathcal{A}_2^{E\cap A_G, R_3})\right)$ because $E \cap A_G \in \mathcal{E}_{st}(\mathcal{G})$ is already shown; finally, theorem 2.1 justifies $(E \cap A_G) \cup (E \cap (A_H \backslash A_G)) = E \in \mathcal{E}_{st}(\mathcal{F} \cup \mathcal{H})$; the other way around ($E \in \mathcal{E}_{st}(\mathcal{G} \cup \mathcal{H})$ implies $E \in \mathcal{E}_{st}(\mathcal{F} \cup \mathcal{H})$) is similar; 2^{nd} case: suppose $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G}) = \emptyset$; then, $\mathcal{E}_{st}(\mathcal{F} \cup \mathcal{H}) = \mathcal{E}_{st}(\mathcal{G} \cup \mathcal{H}) = \emptyset$ holds because assuming $E \in \mathcal{E}_{st}(\mathcal{F} \cup \mathcal{H})$ yields to $E \cap A_F \in \mathcal{E}_{st}(\mathcal{F})$ (splitting theorem 2.1) which contradicts $\mathcal{E}_{st}(\mathcal{F}) = \emptyset$; hence, $E \in \mathcal{E}_{st}(\mathcal{F} \cup \mathcal{H})$ implies $E \in \mathcal{E}_{st}(\mathcal{G} \cup \mathcal{H})$ (as well as the converse direction)
($\Rightarrow$) we will show the contrapositive that is: $(\mathcal{E}_{st}(\mathcal{F}) \neq \mathcal{E}_{st}(\mathcal{G}) \vee A_F \neq A_G) \wedge (\mathcal{E}_{st}(\mathcal{F}) \neq \mathcal{E}_{st}(\mathcal{G}) \vee \mathcal{E}_{st}(\mathcal{G}) \neq \emptyset)$ implies the existence of an AF $\mathcal{H}$, s.t. $\mathcal{E}_{st}(\mathcal{F} \cup \mathcal{H}) \neq \mathcal{E}_{st}(\mathcal{G} \cup \mathcal{H})$ holds; 1. given $\mathcal{E}_{st}(\mathcal{F}) \neq \mathcal{E}_{st}(\mathcal{G})$, then w.l.o.g. we may assume the existence of an $E : E \in \mathcal{E}_{st}(\mathcal{F}) \wedge E \notin \mathcal{E}_{st}(\mathcal{G})$; consider now $\mathcal{H} = (\{a\}, \emptyset)$ whereas $\{a\} \cap (A_F \cup A_G) = \emptyset$ holds; we immediately derive that $E \cup \{a\}$ is a stable extension of $\mathcal{F} \cup \mathcal{H}$; furthermore it is impossible that $E \cup \{a\} \in \mathcal{E}_{st}(\mathcal{G} \cup \mathcal{H})$ holds since $E \notin \mathcal{E}_{st}(\mathcal{G})$ was assumed (splitting theorem 2.2); 2. assume now $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G}) \wedge \mathcal{E}_{st}(\mathcal{G}) \neq \emptyset \wedge A_F \neq A_G$; w.l.o.g. let a be an argument, s.t. $a \in A_F \backslash A_G$ holds; we observe for all stable extensions E of $\mathcal{F}$ and $\mathcal{G}$, $a \notin E$; since $\mathcal{E}_{st}(\mathcal{G}) \neq \emptyset$ was supposed we may assume $E \in \mathcal{E}_{st}(\mathcal{G})$; define $\mathcal{H} = (\{a\}, \emptyset)$, hence $E' = E \cup \{a\} \in \mathcal{E}_{st}(\mathcal{G} \cup \mathcal{H})$ (splitting theorem 2.1); on the other hand $E' \notin \mathcal{E}_{st}(\mathcal{F} \cup \mathcal{H})$ since $\mathcal{F} = \mathcal{F} \cup \mathcal{H}$ and $a \in E'$ holds □

It obviously holds that strong equivalence between two AFs $\mathcal{F}$ and $\mathcal{G}$ implies their weak expansion equivalence. The following example demonstrates that the converse does not hold.

Example 4. Given $\mathcal{F} = (\{a_1, a_2, a_3\}, \{(a_1, a_2), (a_1, a_3)\})$,
$\mathcal{G} = (\{a_1, a_2, a_3\}, \{(a_1, a_2), (a_1, a_3), (a_2, a_3)\})$ and $\mathcal{H} = (\{a_1, a_2\}, \{(a_2, a_1)\})$.

Obviously we have $\mathcal{F} \equiv^{st}_{\prec^N_W} \mathcal{G}$ since $A_F = A_G$ and $\mathcal{E}_{st}(\mathcal{F}) = \mathcal{E}_{st}(\mathcal{G}) = \{a_1\}$ holds. On the other hand we have $\mathcal{E}_{st}(\mathcal{F} \cup \mathcal{H}) = \{\{a_1\}, \{a_2, a_3\}\}$ and $\mathcal{E}_{st}(\mathcal{G} \cup \mathcal{H}) = \{\{a_1\}, \{a_2\}\}$. Hence they are not strong equivalent.

6 Related Work and Conclusions

In this paper, we provided splitting results for Dung-style AFs under the most important semantics, namely stable, preferred, complete and grounded semantics. In a nutshell, the results show that each extension E of a splitted argumentation framework $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2, R_3)$ is equal to the union of an extension E_1 of $\mathcal{A}_1$ and an extension E_2 of a modification (w.r.t. E_1 and R_3) of $\mathcal{A}_2$.

In [10] Baroni et al. introduced a general recursive schema for argumentation semantics. Furthermore they have shown that all admissibility-based semantics are covered by this definition. The great benefit of this approach is that the extensions of an AF $\mathcal{A}$ can be incrementally constructed by the extensions along its strongly connected components.

A directed graph is strongly connected if there is a path from each vertex to every other vertex. The SCCs of a graph are its maximal strongly connected subgraphs. Contracting every SCC to a single vertex leads to an acyclic graph. Hence every SCC-decomposition can be easily transformed into a splitting[6]. Conversely, a given splitting $(\mathcal{A}_1, \mathcal{A}_2, R_3)$ simplifies the computation of SCCs because every SSC is either in $\mathcal{A}_1$ or $\mathcal{A}_2$. In this sense our results are certainly related to the SCC-approach. However, there are some important differences at various levels:

1. there is a subtle, yet relevant difference on the technical level: whereas the approach of Baroni et al. is based on a **generalized** theory of abstract argumentation (see subsection 5.1 in [10]), we stick to Dung's original approach and use an adequate modification in addition to the reduct to establish our results;
2. we provide theoretical insights about the relationship of the extensions of an **arbitrary** splitted AF; the parts into which an AF is split may be, but do not necessarily have to be SCCs;
3. whereas a major motivation in [10] was the identification of new semantics satisfying SCC-recursiveness, our primary intent is to carry over our results to **dynamical argumentation** like new terms of equivalence.

In section 4 we illustrated how to carry over our splitting results to dynamical argumentation. A number of papers appeared in this field of research. However, the possibility of reusing extensions has not received that much attention yet. A mentionable work in this context is [4]. Cayrol et al. proposed a typology of revisions (one new argument, one new interaction). Furthermore they proved sufficient conditions for being a certain revision type.

In future work we would like to study in detail the mentioned terms of equivalence between two AFs $\mathcal{A}$ and $\mathcal{B}$, i.e. what are sufficient and necessary conditions for their weak (strong, normal) expansion equivalence w.r.t. a semantics σ.

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[6] Take the union of the initial nodes of the decomposition ($= \mathcal{A}_1$) and the union of the remaining subgraph ($= \mathcal{A}_2$).

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