Normal and Strong Expansion Equivalence for Argumentation Frameworks

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Abstract

Given a semantics σ, two argumentation frameworks (AFs) $\mathcal{F}$ and $\mathcal{G}$ are said to be *standard equivalent* if they possess the same extensions and *strongly equivalent* if, for any AF $\mathcal{H}$, $\mathcal{F}$ conjoined with $\mathcal{H}$ and $\mathcal{G}$ conjoined with $\mathcal{H}$ are standard equivalent. Argumentation is a dynamic process and, in general, new arguments occur in response to a former argument or, more precisely, attack a former argument. For this reason, rather than considering arbitrary expansions we focus here on expansions where new arguments and attacks may be added but the attacks among the old arguments remain unchanged. We define and characterize two new notions of equivalence between AFs (which lie in-between standard and strong equivalence), namely *normal* and *strong expansion equivalence*. Furthermore, using the characterization theorems proved in this paper, we draw the connections between all mentioned notions of equivalence including further equivalence relations, so-called *weak* and *local expansion equivalence*.

Keywords: abstract argumentation, meta-properties, equivalence relations

1. Introduction

In the last two decades argumentation theory has received considerable interest in the AI-community (a basic overview is given in [1]). Two main directions to handle argumentation formally have been put forward in the literature. First, so-called *abstract argumentation*. It is mainly concerned with handling or resolving conflicts among arguments without considering their internal structure. The most known and extensively studied abstract system is the one proposed by Dung in [2]. Dung-style argumentation frameworks (AFs) are simply digraphs treating arguments and attacks as undefined primitives represented by vertices or edges, respectively. A variety of semantics is provided. Each of them captures different intuitions about how to reason over conflicting knowledge. Reasoning in each case is non-monotonic, that is, new arguments may attack older ones and thus lead to the rejection of formerly accepted arguments.

The second approach is *deductive argumentation* (a comprehensive overview can be found in [3]) which is concerned with constructing arguments from a given knowledge base and furthermore, determining their strength by applying a reasonable defined notion of attack. The common interaction between both is that deductive AFs instantiate abstract AFs. There are two mentionable results which show that reasonable definitions and concepts on the deductive and abstract level may yield undesired and unintuitive results if they are linked. Besnard and Hunter [4] showed that conveying a deductive defeat-relation to Dung's abstract AFs in a straightforward manner may cause a collapse of several semantics. Even worse, Caminada and Amgoud [5] showed that the outcome of instantiated AFs may fail to satisfy very basic requirements like consistency. In order to avoid such anomalous results they introduced so-called *rationality postulates* which should be satisfied by any deductive system. In summary, solution concepts on the abstract level do not necessarily make sense in consideration of deductive arguments. This means, the study of properties and concepts of argumentation on the abstract level should be driven by reasonable instantiations.

The work presented in this paper is part of the abstract approach to argumentation. It focuses on new notions of equivalence for abstract argumentation frameworks. In general, equivalence tells us whether two syntactically different knowledge bases represent the same information - which is relevant, for instance, when we want to find out whether a knowledge base can be replaced by a simpler one. The standard, model based notion of equivalence is appropriate for monotonic logics, yet in the light of potential augmentations stronger notions are necessary for nonmonotonic formalisms. Take as an example logic programs under stable model semantics [6]. The two programs $P_1 = \{a\}$ and $P_2 = \{a \leftarrow \mathit{not}\ b\}$ have the same stable model, namely $\{a\}$. However, if P_1 and P_2 are later extended with the fact b, then the stable models will no longer coincide: we obtain $\{a, b\}$ for the former, $\{b\}$ for the latter.

This observation led to the investigation of stronger equivalence notions for logic programs, and more recently also for argumentation. Oikarinen and Woltran [7] introduced the notion of *strong equivalence* for abstract AFs. Two AFs $\mathcal{F}$ and $\mathcal{G}$ are strongly equivalent if for any AF $\mathcal{H}$, $\mathcal{F}$ conjoined with $\mathcal{H}$ and $\mathcal{G}$ conjoined with $\mathcal{H}$ possess the same extensions. This powerful notion of equivalence is the starting point for our research. However, for several typical argumentation scenarios strong equivalence seems too strong a notion. Just like in the case of other non-monotonic formalisms where further equivalence notions in-between strong and standard equivalence were motivated, defined and studied (see [8] for an excellent overview) we looked for corresponding notions for AFs which take the very nature of argumentation into account.

What we have in mind can be illustrated with a citation [9]:

> How does argumentation usually take place? Argumentation starts when an initial argument is put forward, making some claim. An objection is raised, in the form of a counterargument. The latter is addressed in turn, eventually giving rise to a counter-counterargument, if any. And so on.

These specific kinds of dynamic argumentation scenarios are the focus of our study in this paper. Let us consider a reasoning process about defeasible information stored in a knowledge base. What happens on the abstract level if a new piece of information is added? It turns out that in almost all deductive argumentation systems older arguments and their corresponding attacks survive and only new arguments which may interact with the previous ones arise (compare [10]). This means, in contrast to Oikarinen and Woltran which studied equivalence w.r.t. to arbitrary expansions we are interested in equivalence relations w.r.t. specific expansions where the attack relationship between former arguments remains unchanged. Such kinds of dynamic scenarios correspond with the already defined concepts of *normal* or *strong expansions* [11].

Let us consider Dung-style AFs from another point of view - not necessarily connected with the field of argumentation described above. Without doubt, AFs are among the simplest non-monotonic systems one can think of. Yet, this approach is still powerful. It can be seen as a general theory capturing several non-monotonic formalisms as well as a tool for solving well-known problems as the stable-marriage problem [2]. The investigation of meta-properties like splitting results [12, 13], replacement theorems [7, 14] and intertranslability results [15] has begun quite recently and is still at the beginning. The *logical approach*, a methodology introduced by David Pearce [16], represents the view that logical or meta-logical analysis of, for example, non-monotonic formalisms can be a source of inspiration and may help in the analysis of certain (practical) problems and the development of the formal framework. In this sense, we believe that our fine grained analysis of equivalence relations which allow inter-substitutability in certain dynamic contexts will be even fruitful for practical problems like it was shown in the case of splitting results and its positive influence on the computational complexity [17].

The main contributions and organization of the paper are as follows. Section 2 reviews the necessary definitions in abstract argumentation including several argumentation semantics, notions of expansion as well as former splitting results [12] which will be used as a tool for simplifying proofs. In Section 3, we present several notions of equivalence containing existing characterization theorems w.r.t. strong equivalence [7]. Furthermore, we draw some preliminary relations between all introduced equivalence relations. The main results, i.e. characterization theorems for strong and normal expansion equivalence w.r.t. stable, semi-stable, admissible, preferred, ideal, grounded and complete semantics, are then contained in the following two sections. In particular, quite surprisingly, strong expansion equivalence coincides with strong equivalence in case of stable and semi-stable semantics. This result does not hold for the other semantics considered in this paper. This means, in case of admissible, preferred, ideal, grounded and complete semantics strong expansion equivalence is weaker than strong equivalence. As a further unexpected result we showed that for any considered semantics, two AFs are strongly equivalent if and only if they are normal expansion equivalent. Section 6 summarizes the results and provides some additional observations. In particular, we discuss the role of self-loop-free AFs where normal and strong expansion equivalence collapse to syntactical identity. Furthermore, we use our new characterization results to draw a full picture how the different equivalence relations are related. Finally, in Section 7 we conclude and discuss related work.

2. Preliminaries

An *argumentation framework (AF)* is a pair $\mathcal{F} = (A, R)$, where A is a non-empty (possibly infinite) set whose elements are called *arguments* and $R \subseteq A \times A$ a binary relation, called the *attack relation*. In this paper we restrict ourselves to finite AFs. If $(a, b) \in R$ holds we say that *a attacks* b, or b is *defeated* by a in $\mathcal{F}$. We will slightly abuse notations, and write $(A, b) \in R$ for $\exists a \in A : (a, b) \in R$; likewise we use $(b, A) \in R$ and $(A, A') \in R$. An argument $a \in A$ is *defended* by a set $A' \subseteq A$ in $\mathcal{F}$ if for each $b \in A$ with $(b, a) \in R$, $(A', b) \in R$. Furthermore, we say that a set $A' \subseteq A$ is *conflict-free* in $\mathcal{F}$ if there are no arguments $a, b \in A'$ such that a attacks b. The set of all conflict-free sets of an AF $\mathcal{F}$ is denoted by $cf(\mathcal{F})$. For an AF $\mathcal{F} = (B, S)$ we use $A(\mathcal{F})$ to refer to B and $R(\mathcal{F})$ to refer to S. Finally, we introduce the union for two AFs $\mathcal{F}$ and $\mathcal{G}$ as $\mathcal{F} \cup \mathcal{G} = (A(\mathcal{F}) \cup A(\mathcal{G}), R(\mathcal{F}) \cup R(\mathcal{G}))$.

2.1. Extension-based Semantics

A semantics σ specifies criteria for determining, given an AF $\mathcal{F}$, sets of arguments where each one of them is considered to be acceptable w.r.t. $\mathcal{F}$. These sets are called *σ-extensions* or if clear from context *extensions* of $\mathcal{F}$. The set of all extensions is denoted by $\mathcal{E}_\sigma(\mathcal{F})$. One minimal requirement which all existing semantics have in common, is that an extension has to be conflict-free. We consider here the classical Dung semantics, namely, stable, admissible, preferred, complete, grounded as well as the ideal and semi-stable semantics [2, 18, 19].

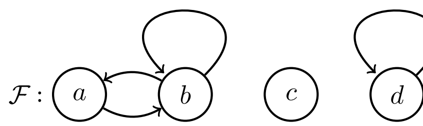
Definition 1. Let $\mathcal{F} = (A, R)$ be an AF and $E \subseteq A$. E is a

1. *stable extension* ($E \in \mathcal{E}_{st}(\mathcal{F})$) iff
   $E \in cf(\mathcal{F})$ and for every $a \in A \backslash E$, $(E, a) \in R$ holds,
2. *admissible extension*[1] ($E \in \mathcal{E}_{ad}(\mathcal{F})$) iff
   $E \in cf(\mathcal{F})$ and each $a \in E$ is defended by E in $\mathcal{A}$,
3. *preferred extension* ($E \in \mathcal{E}_{pr}(\mathcal{F})$) iff
   $E \in \mathcal{E}_{ad}(\mathcal{F})$ and for each $E' \in \mathcal{E}_{ad}(\mathcal{F})$, $E \not\subset E'$ holds,
4. *complete extension* ($E \in \mathcal{E}_{co}(\mathcal{F})$) iff
   $E \in \mathcal{E}_{ad}(\mathcal{F})$ and for each $a \in A$ defended by E in $\mathcal{F}$, $a \in E$ holds,
5. *grounded extension* ($E \in \mathcal{E}_{gr}(\mathcal{F})$) iff
   $E \in \mathcal{E}_{co}(\mathcal{F})$ and for each $E' \in \mathcal{E}_{co}(\mathcal{F})$, $E' \not\subset E$ holds,
6. *ideal extension* of $\mathcal{F}$ ($E \in \mathcal{E}_{id}(\mathcal{F})$) iff
   $E \in \mathcal{E}_{ad}(\mathcal{F})$, $E \subseteq \bigcap_{P \in \mathcal{E}_{pr}(\mathcal{F})} P$ and for each $A \in \mathcal{E}_{ad}(\mathcal{F})$ with the property $A \subseteq \bigcap_{P \in \mathcal{E}_{pr}(\mathcal{F})} P$, $E \not\subset A$ holds,

[1] Note that it is more common to speak about admissible sets instead of the admissible extensions. For reasons of unified notation we used the uncommon version.

7. *semi-stable extension* ($E \in \mathcal{E}_{ss}(\mathcal{F})$) iff
$E \in \mathcal{E}_{ad}(\mathcal{F})$ and for each $E' \in \mathcal{E}_{ad}(\mathcal{F})$, $R^+_{\mathcal{F}}(E) \not\subset R^+_{\mathcal{F}}(E')$ holds, where $R^+_{\mathcal{F}}(E) = E \cup \{b \mid (a,b) \in R, a \in E\}$.

Example 1. Let $\mathcal{F} = (\{a,b,c,d\}, \{(a,b),(b,a),(b,b),(d,d)\})$. The graph representation of $\mathcal{F}$ is given as follows.



We observe that there are four conflict-free sets, namely $\emptyset$, $\{a\}$, $\{c\}$ and $\{a,c\}$. The following table illustrates whether these sets are σ-extensions (denoted by $\times$) or not.

	st	*ss*	*pr*	*co*	*ad*	*gr*	*id*
$\emptyset$					$\times$		
$\{a\}$					$\times$		
$\{c\}$				$\times$	$\times$	$\times$	
$\{a,c\}$		$\times$	$\times$	$\times$	$\times$		$\times$

There are several relations between the mentioned semantics, e.g. for any AF $\mathcal{F}$, $\mathcal{E}_{st}(\mathcal{F}) \subseteq \mathcal{E}_{ss}(\mathcal{F}) \subseteq \mathcal{E}_{pr}(\mathcal{F}) \subseteq \mathcal{E}_{co}(\mathcal{F}) \subseteq \mathcal{E}_{ad}(\mathcal{F})$. The table above shows that stable extensions do not necessarily exist. The semi-stable semantics overcomes this weaknesses and warrants the existence of extensions in case of finite AFs and even for subclasses of infinite AFs [20]. However, the other semantics in question are *universal*, i.e. they always warrant at least one extension. Furthermore, ideal and grounded semantics follow the *unique status approach*, i.e. for any AF $\mathcal{F}$, $|\mathcal{E}_{gr}(\mathcal{F})| = |\mathcal{E}_{id}(\mathcal{F})| = 1$. When regarding the above table we observe that preferred, semi-stable and ideal extensions coincide. It can be shown that this observation holds in general if the considered AF possesses a unique preferred extension (compare proof of Lemma 2 in [21]). This property is stated in the following proposition and will be used frequently throughout the paper.

Proposition 1. *For any AF $\mathcal{F}$, if $|\mathcal{E}_{pr}(\mathcal{F})| = 1$, then $\mathcal{E}_{pr}(\mathcal{F}) = \mathcal{E}_{ss}(\mathcal{F}) = \mathcal{E}_{id}(\mathcal{F})$.*

2.2. Notions of Expansions

As pointed out in the introductory part of this paper we would like to study dynamic notions of equivalence compatible with the very nature of a dispute where new arguments are put forward in response to former arguments. These kinds of dynamic scenarios perfectly fit together with the formal concepts of *normal* and *strong expansions* firstly defined in [11].

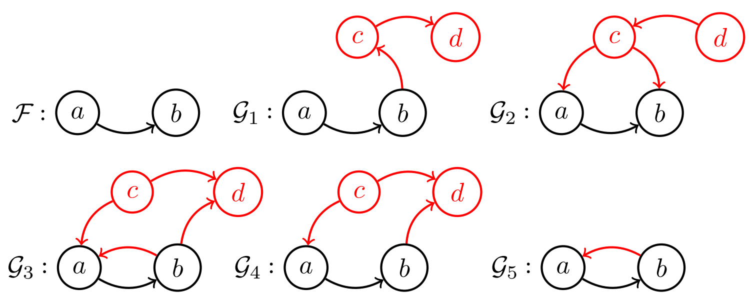
Definition 2. An AF $\mathcal{F}^*$ is an *expansion* of AF $\mathcal{F} = (A, R)$ (for short, $\mathcal{F} \prec \mathcal{F}^*$) iff $\mathcal{F}^*$ has a representation as $(A \cup A^*, R \cup R^*)$, s.t. at least one of A^* and R^* is not empty and $A^* \cap A = R^* \cap R = \varnothing$ holds. Such an expansion is called

1. *normal* ($\mathcal{F} \prec^N \mathcal{F}^*$) iff $A^* \neq \varnothing \wedge \forall ab\ ((a,b) \in R^* \to a \in A^* \vee b \in A^*)$,
2. *strong* ($\mathcal{F} \prec^N_S \mathcal{F}^*$) iff $\mathcal{A} \prec^N \mathcal{A}^*$ and $\forall ab\ ((a,b) \in R^* \to \neg(a \in A \wedge b \in A^*))$,
3. *weak* ($\mathcal{F} \prec^N_W \mathcal{F}^*$) iff $\mathcal{A} \prec^N \mathcal{A}^*$ and $\forall ab\ ((a,b) \in R^* \to \neg(a \in A^* \wedge b \in A))$,
4. *local* ($\mathcal{F} \prec^L \mathcal{F}^*$) iff $A^* = \varnothing$.

For short, normal expansions add new arguments and possibly new attacks. The latter do not contain attacks between previous arguments. Strong (weak) expansions are normal and only add *strong (weak) arguments*, i.e. the added arguments are never attacked by (attack) former arguments. Local expansions do not introduce any new arguments. They only add further attacks between existing arguments. In consideration of the definitions we observe that local expansions are the orthogonal concept to normal expansions or, in other words, any expansions can be split into a normal and local part (compare AFs $\mathcal{G}_3$, $\mathcal{G}_4$ and $\mathcal{G}_5$ in Example 2).

As usual we use $\mathcal{F} \preceq \mathcal{F}^*$ to indicate that the equality case is included, i.e. $\mathcal{F} \prec \mathcal{F}^* \vee \mathcal{F} = \mathcal{F}^*$ holds. The same applies to the other kinds of expansions. The following figures exemplify the definitions.

Example 2. The AF $\mathcal{F}$ is the initial framework. Weak and strong expansions of $\mathcal{F}$ are given by $\mathcal{G}_1$ or $\mathcal{G}_2$, respectively. Furthermore, the AFs $\mathcal{G}_3$, $\mathcal{G}_4$ and $\mathcal{G}_5$ show an arbitrary, normal and local expansion of $\mathcal{F}$.

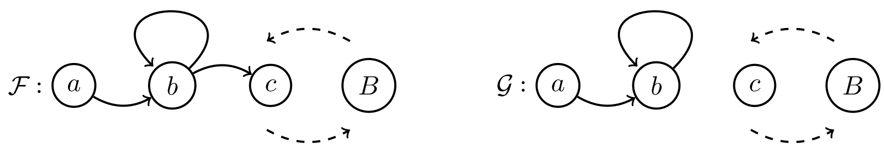


2.3. Splitting Results

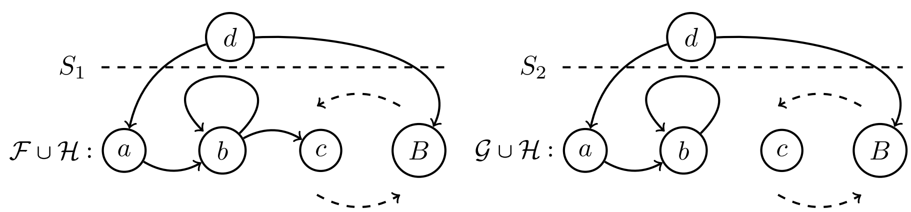
Splitting results are concerned with the question whether it is possible to divide a formal theory T in disjoint subtheories $S_1,\ldots,S_n$ such that the formal semantics of the entire theory T can be obtained by constructing the semantics of $S_1,\ldots,S_n$. Such results, especially in non-monotonic formalisms [22, 23, 24, 12], are of great importance since first, they allow for simplification of proofs showing properties of a particular formalism and second, they may yield more efficient computations. In this paper we will use splitting results for AFs as a tool for simplification.

An ongoing task in this paper is the question whether it is possible to find an AF $\mathcal{H}$ such that, given two AFs $\mathcal{F}$ and $\mathcal{G}$, the semantics of $\mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \cup \mathcal{H}$ do not coincide. The difficulty is that we usually have very limited information about the AFs $\mathcal{F}$ and $\mathcal{G}$.

Example 3. Consider the following AFs $\mathcal{F}$ and $\mathcal{G}$. We have $A(\mathcal{F}) = A(\mathcal{G}) = \{a,b,c\} \cup B$ where B is a (possibly empty) set of further arguments. Furthermore, $R(\mathcal{F}) = \{(a,b),(b,b),(b,c)\} \cup R$ and $R(\mathcal{G}) = \{(a,b),(b,b)\} \cup S$ where R and S represent possible but unknown attacks (indicated by dashed arrows).



Since we have only partial information about the AFs we cannot compute/compare their extensions. For instance, in case of $B = \varnothing$ we deduce $\mathcal{E}_{pr}(\mathcal{F}) = \mathcal{E}_{pr}(\mathcal{G}) = \{\{a,c\}\}$, i.e. they possess the same preferred extension. Consider now the AFs $\mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \cup \mathcal{H}$ where $\mathcal{H} = (A(\mathcal{F}) \cup \{d\}, \{(d,a)\} \cup \{(d,b) \mid b \in B\})$. Observe that $\mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \cup \mathcal{H}$ are strong expansions of $\mathcal{F}$ or $\mathcal{G}$, respectively.



Using splitting results for AFs (consider [12] for detailed information) we are able to compute the extensions of the partially known AFs $\mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \cup \mathcal{H}$ in an iterative way. The procedure is as follows: We split the initial AFs into two subframeworks, namely $\mathcal{F}_1 = \mathcal{G}_1 = (\{d\}, \varnothing)$ and $\mathcal{F}_2 = \mathcal{F}$ or $\mathcal{G}_2 = \mathcal{G}$, respectively (indicated by the dashed lines S_1 and S_2). We then take the unique preferred extension $\{d\}$ of $\mathcal{F}_1$ and $\mathcal{G}_1$ to reduce the AFs $\mathcal{F}_2$ or $\mathcal{G}_2$, respectively. In this case, "reducing" quite simply means deleting all arguments attacked by d. The following AFs $\mathcal{F}_2^{\{d\}}$ and $\mathcal{G}_2^{\{d\}}$ illustrate these reducts.

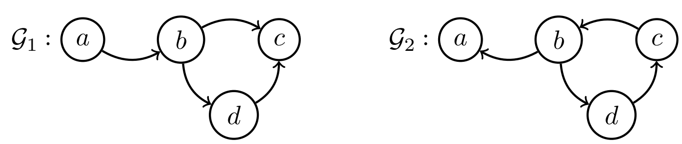
We now compute the preferred extensions of the reduced AFs, namely $\mathcal{E}_{pr}(\mathcal{F}_2^{\{d\}}) = \{\emptyset\}$ and $\mathcal{E}_{pr}(\mathcal{G}_2^{\{d\}}) = \{\{c\}\}$. Finally, the preferred extensions of $\mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \cup \mathcal{H}$ can be obtained by combining $\{d\}$ and $\emptyset$ or $\{c\}$, respectively. That means, $\{\{d\}\} = \mathcal{E}_{pr}(\mathcal{F} \cup \mathcal{H}) \neq \mathcal{E}_{pr}(\mathcal{G} \cup \mathcal{H}) = \{\{c, d\}\}$. In summary, we have shown that the partially known AFs $\mathcal{F}$ and $\mathcal{G}$ have strong expansions which possess different preferred and hence, semi-stable and ideal extensions (Prop. 1).

3. Notions of Equivalence

It is well-known that logically equivalence in propositional or first order logic is even a congruence relation w.r.t. the logical connectives. This property is the main reason for the validity of the so-called *replacement theorem* which states that if two formulae ϕ_1 and ϕ_2 are logically equivalent then no change in the set of models of any formula Φ occurs if we replace one of them with the other (compare Theorem 4.1 in [25]). For short, possessing the same models guarantees inter-substitutability in any logical context. Unfortunately, the analogous statement in case of abstract AFs (as well as in other non-monotonic formalisms) does not hold. Consider the following AFs.

Example 4. The AFs $\mathcal{F}_1$ and $\mathcal{F}_2$ possess the unique preferred extension $\{a, c\}$.

The AF $\mathcal{G}_2$ syntactically results by replacing the subframework $\mathcal{F}_1$ of $\mathcal{G}_1$ with $\mathcal{F}_2$. Observe that $\mathcal{E}_{pr}(\mathcal{G}_1) = \{\{a, d\}\} \neq \{\emptyset\} = \mathcal{E}_{pr}(\mathcal{G}_2)$.



We now introduce several notions of equivalence between AFs. We will point out some prelimary relations between them and illustrate their usefulness for certain kinds of dynamics.

3.1. Standard Equivalence

The simplest concept of equivalence between two AFs is to have the same extensions. This equivalence relation corresponds to a non-dynamical, static argumentation scenario. All queries w.r.t. credulous or skeptical accepted arguments are answered identically. In this sense both are mutually replaceable.

Definition 3. Two AFs $\mathcal{F}$ and $\mathcal{G}$ are *(standard) equivalent* to each other w.r.t. a semantics σ, in symbols $\mathcal{F} \equiv^\sigma \mathcal{G}$, iff $\mathcal{E}_\sigma(\mathcal{F}) = \mathcal{E}_\sigma(\mathcal{G})$ holds.

We recall some relations concerning standard equivalence and different semantics which will be used throughout the paper. Quite surprisingly, none of the other implications hold as shown in [7].

Proposition 2 (Prop. 1 in [7]). *For any AFs $\mathcal{F}$ and $\mathcal{G}$,*

1. $\mathcal{F} \equiv^{ad} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{\sigma} \mathcal{G}$, $\sigma \in \{pr, id\}$,
2. $\mathcal{F} \equiv^{co} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{\sigma} \mathcal{G}$, $\sigma \in \{pr, gr, id\}$.

3.2. Strong Equivalence

Standard equivalence of two AFs is not sufficient for their mutual replaceability in dynamic argumentation scenarios. That means, possessing the same extensions does not guarantee to share the same acceptable sets of arguments w.r.t. all expansions as illustrated in Example 4. Oikarinen and Woltran therefore defined an equivalence relation which enforces this property, so-called *strong equivalence* [7].

Definition 4. Two AFs $\mathcal{F}$ and $\mathcal{G}$ are *strongly equivalent* to each other w.r.t. a semantics σ, in symbols $\mathcal{F} \equiv^{\sigma}_{\leq} \mathcal{G}$, iff for each AF $\mathcal{H}$, $\mathcal{F} \cup \mathcal{H} \equiv^{\sigma} \mathcal{G} \cup \mathcal{H}$ holds.[2]

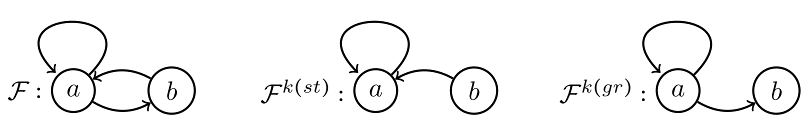
Deciding strong equivalence w.r.t. a semantics σ between two AFs $\mathcal{F}$ and $\mathcal{G}$ can be done by comparing their corresponding *σ-kernels* $\mathcal{F}^{k(\sigma)}$ and $\mathcal{G}^{k(\sigma)}$. A σ-kernel of an AF $\mathcal{F}$ is itself an AF obtained from $\mathcal{F}$ by deleting certain attacks depending on the considered semantics σ.

Definition 5. Given a semantics $\sigma \in \{st, ad, gr, co\}$ and an AF $\mathcal{F} = (A, R)$. We define the σ-kernel of $\mathcal{F}$ as $\mathcal{F}^{k(\sigma)} = (A, R^{k(\sigma)})$ where

1. $R^{k(st)} = R \setminus \{(a,b) \mid a \neq b, (a,a) \in R\}$,
2. $R^{k(ad)} = R \setminus \{(a,b) \mid a \neq b, (a,a) \in R, \{(b,a),(b,b)\} \cap R \neq \emptyset\}$,
3. $R^{k(gr)} = R \setminus \{(a,b) \mid a \neq b, (b,b) \in R, \{(a,a),(b,a)\} \cap R \neq \emptyset\}$,
4. $R^{k(co)} = R \setminus \{(a,b) \mid a \neq b, (a,a),(b,b) \in R\}$.

We want to mention three simple properties applying to any σ-kernel defined above: First, $\mathcal{F}$ and $\mathcal{F}^{k(\sigma)}$ share exactly the same arguments. Second, the attack-relation of $\mathcal{F}^{k(\sigma)}$ is contained in the attack-relation of $\mathcal{F}$ and third, the kernel operation is idempotent, i.e. $\mathcal{F}^{k(\sigma)} = \left(\mathcal{F}^{k(\sigma)}\right)^{k(\sigma)}$.

Example 5. Consider the AF $\mathcal{F}$ and its corresponding stable-kernel $\mathcal{F}^{k(st)}$ and grounded-kernel $\mathcal{F}^{k(gr)}$.



[2]In order to have a uniform notation we slightly differ here from the original version "$\mathcal{F} \equiv^{\sigma}_{s} \mathcal{G}$". Def. 2 justifies the replacement of "s" by "$\leq$".

We list now some non-trivial results (taken from [7][3]) showing relations between the syntactical concept of σ-kernels and semantical defined equivalence relations.

Lemma 1. *For any AF $\mathcal{F}$ and $\sigma \in \{st, ad, gr, co\}$, $\mathcal{F} \equiv^{\sigma} \mathcal{F}^{k(\sigma)}$.*

Lemma 2. *For any AFs $\mathcal{F}$, $\mathcal{G}$ and $\sigma \in \{st, ad, gr, co\}$ the following holds: If $\mathcal{F}^{k(\sigma)} = \mathcal{G}^{k(\sigma)}$, then $(\mathcal{F} \cup \mathcal{H})^{k(\sigma)} = (\mathcal{G} \cup \mathcal{H})^{k(\sigma)}$ for all AFs $\mathcal{H}$.*

Theorem 1. *For any AFs $\mathcal{F}$, $\mathcal{G}$ and $\sigma \in \{st, ad, gr, co\}$:*

$$\mathcal{F}^{k(\sigma)} = \mathcal{G}^{k(\sigma)} \Leftrightarrow \mathcal{F} \equiv^{\sigma}_{\preceq} \mathcal{G}.$$

Theorem 2. *For any AFs $\mathcal{F}$ and $\mathcal{G}$:*

$$\mathcal{F}^{k(co)} = \mathcal{G}^{k(co)} \Leftrightarrow \mathcal{F}^{k(ad)} = \mathcal{G}^{k(ad)} \text{ and } \mathcal{F}^{k(gr)} = \mathcal{G}^{k(gr)}.$$

Theorem 3. *For any AFs $\mathcal{F}$ and $\mathcal{G}$ the following holds:*

$$\mathcal{F} \equiv^{ad}_{\preceq} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv^{pr}_{\preceq} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv^{id}_{\preceq} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv^{ss}_{\preceq} \mathcal{G}.$$

Consider Example 5. In case of stable semantics we may delete attacks from self-attacking arguments and still retain the same extensions ($\mathcal{F} \equiv^{st} \mathcal{F}^{k(st)}$, Lemma 1). Due to the idempotency of *st*-kernel even the same extensional behavior w.r.t. arbitrary expansions is guaranteed ($\mathcal{F} \equiv^{st}_{\preceq} \mathcal{F}^{k(st)}$, Theorem 1). In case of grounded semantics the conditions under which an attack (a, b) is redundant in the light of dynamics differ, namely: First, a and b are self-attacking or second, if b is self-defeating and counterattacks a. The latter is the case in Example 5.

3.3. Normal, Strong, Weak and Local Expansion Equivalence

Now we turn to four intermediate forms of equivalence falling between the aforementioned standard and strong equivalence. We start with their formal definitions.

Definition 6. Given a semantics σ. Two AFs $\mathcal{F}$ and $\mathcal{G}$ are

1. *normal expansion equivalent* w.r.t. σ, in symbols $\mathcal{F} \equiv^{\sigma}_{\preceq^N} \mathcal{G}$, iff for each AF $\mathcal{H}$, s.t. $\mathcal{F} \preceq^N \mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \preceq^N \mathcal{G} \cup \mathcal{H}$, $\mathcal{F} \cup \mathcal{H} \equiv^{\sigma} \mathcal{G} \cup \mathcal{H}$ holds,
2. *strong expansion equivalent* w.r.t. σ, in symbols $\mathcal{F} \equiv^{\sigma}_{\preceq^N_S} \mathcal{G}$, iff for each AF $\mathcal{H}$, s.t. $\mathcal{F} \preceq^N_S \mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \preceq^N_S \mathcal{G} \cup \mathcal{H}$, $\mathcal{F} \cup \mathcal{H} \equiv^{\sigma} \mathcal{G} \cup \mathcal{H}$ holds,
3. *weak expansion equivalent* w.r.t. σ, in symbols $\mathcal{F} \equiv^{\sigma}_{\preceq^N_W} \mathcal{G}$, iff for each AF $\mathcal{H}$, s.t. $\mathcal{F} \preceq^N_W \mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \preceq^N_W \mathcal{G} \cup \mathcal{H}$, $\mathcal{F} \cup \mathcal{H} \equiv^{\sigma} \mathcal{G} \cup \mathcal{H}$ holds,

[3] We use the shortcut "$x \leftarrow y$" for "x corresponds with or is a summary of y in [7]". Lemma 1 $\leftarrow$ Lemmata 1,4,6,10; Lemma 2 $\leftarrow$ Lemmata 2,5,7,11; Theorem 1 $\leftarrow$ Theorems 1,2,3,4; Theorem 2 $\leftarrow$ Theorem 5; Theorem 3 $\leftarrow$ Theorem 2.

4. *local expansion equivalent*[4] w.r.t. σ, in symbols $\mathcal{F} \equiv^{\sigma}_{\preceq_L} \mathcal{G}$, iff for each AF $\mathcal{H}$, s.t. $A(\mathcal{H}) \subseteq A(\mathcal{F} \cup \mathcal{G})$, $\mathcal{F} \cup \mathcal{H} \equiv^{\sigma} \mathcal{G} \cup \mathcal{H}$ holds.

Normal expansions corresponds with dynamic scenarios where it is assumed that - before adding new arguments - the attack relationships among arguments put forward earlier have been fully clarified and there is no further dispute concerning these relations. Such kinds of dynamics naturally occur (on the abstract level) if a new piece of information is added to the knowledge base (deductive level) provided that the underlying notion of argument and attack is maintained. In consideration of such a (re-)instantiation process normal expansion equivalence allows replacements without loss of information.

Strong expansion equivalence captures the idea of inter-substitutability in context of only adding *stronger* arguments. In consideration of the very nature of a dispute where further arguments occur in response to former arguments this equivalence relation appears as the most important one. Characterizing weak expansion equivalence seems to be more of an academic exercise than a task with practical relevance. Being aware of this fact, we emphasize that there are formalisms, like Value Based AFs [26] where the question of weak expansion equivalence might be relevant. Former arguments may be arguments which advance higher values than the further arguments. Consequently, the new arguments cannot attack the former (compare the idea of "attack-succeed" in [26]).

The last intermediate form of equivalence, namely local expansions equivalence was firstly defined in [7]. The appearance of new attacks between existing arguments may occur if the underlying attack definition is changed and the abstract AF has to re-instantiated. A detailed analysis of attack-relations can be found in [27].

3.4. Preliminary Relations

Now we present some preliminary relations between the mentioned notions of equivalence. The presented implications follow directly from Definitions 3, 4 and 6. Figure 1 summarizes all results in a compact way.

Proposition 3. *For any AFs $\mathcal{F}$, $\mathcal{G}$, and **every** (possible) semantics σ the following holds:*

1. $\mathcal{F} \equiv^{\sigma}_{\preceq} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{\sigma}_{\preceq^N} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{\sigma}_{\preceq^N_S} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{\sigma} \mathcal{G}$
2. $\mathcal{F} \equiv^{\sigma}_{\preceq} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{\sigma}_{\preceq^N} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{\sigma}_{\preceq^N_W} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{\sigma} \mathcal{G}$
3. $\mathcal{F} \equiv^{\sigma}_{\preceq} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{\sigma}_{\preceq_L} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{\sigma} \mathcal{G}$

[4]Oikarinen and Woltran called this *locally (strongly) equivalent* (compare Def. 8 in [7]). Note that the definiens of $\mathcal{F} \equiv^{\sigma}_{\preceq_L} \mathcal{G}$ imply that the considered AFs $\mathcal{H}$ satisfy $\mathcal{F}_{\mathcal{H}} \preceq^L \mathcal{F}_{\mathcal{H}} \cup \mathcal{H}$ and $\mathcal{G}_{\mathcal{H}} \preceq^L \mathcal{F}_{\mathcal{H}} \cup \mathcal{H}$ where $\mathcal{F}_{\mathcal{H}} = (A(\mathcal{F}) \cup A(\mathcal{H}), R(\mathcal{F}))$ and $\mathcal{G}_{\mathcal{H}} = (A(\mathcal{G}) \cup A(\mathcal{H}), R(\mathcal{G}))$.

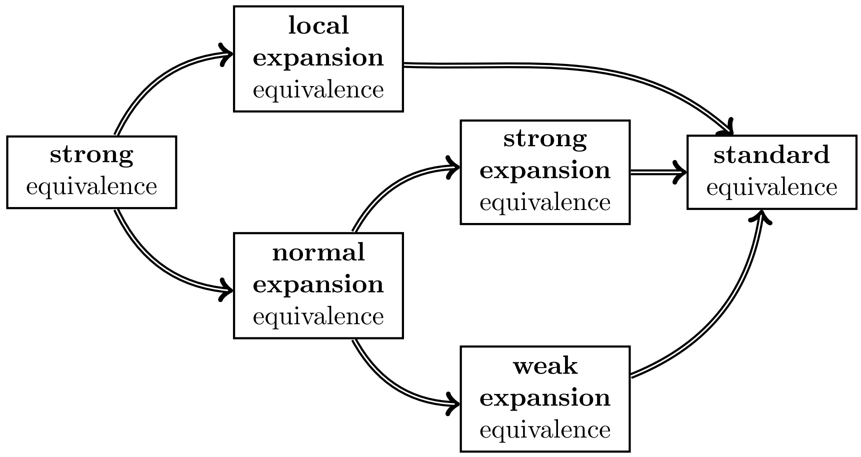


Figure 1: Relations between Equivalence Notions

In Section 6.3 we will consider again the relations between different notions of equivalence. We will use the results proven in the following two sections to strengthen the statements in Proposition 3.

4. Characterizing Strong Expansion Equivalence

In this section we will characterize strong expansion equivalence for stable, semi-stable, admissible, preferred, ideal, grounded and complete semantics. Analogously to the characterization of strong equivalence we provide syntactical criteria to decide this notion of equivalence, so-called σ-*-*kernels*. The main results of this section can be summarized as follows:

- Strong expansion equivalence w.r.t. stable and semi-stable semantics can be decided by the already defined *st*- and *ad*-kernel [7]. This means, in case of these semantics, strong and strong expansion equivalence coincide.
- The concepts of strong expansion equivalence w.r.t. admissible, preferred and ideal semantics coincide and can be adequately described by the newly introduced *ad*-*-kernel. Strong equivalence w.r.t. these semantics implies strong expansion equivalence but not vice versa.
- The characterization of strong expansion equivalence w.r.t. grounded and complete semantics was the most difficult part. These notions can be decided by the newly introduced *gr*- or *co*-*-kernel, respectively. Both concepts are weaker than there corresponding strong equivalence notions.

4.1. Strong Expansion Equivalence for Stable Semantics

Oikarinen and Woltran proved that attacks (a, b) where a is a self-attacking argument do not contribute in the evaluation of an AF $\mathcal{F}$, no matter how $\mathcal{F}$

is extended. Furthermore they showed that the syntactical equivalence of *st*-kernels, which "delete" such attacks of a given AF, is necessary and sufficient for strong equivalence between two AFs (Theorem 1). Since the classes of strong and arbitrary expansions are in a proper subset relation we suspected that there are strong expansion equivalent AFs which are not strongly equivalent. The construction of such an example failed and we tried to prove that even strong expansion equivalence between two AFs is fulfilled if and only if they possess the same *st*-kernels. The following theorem proves this conjecture. Remember that the *st*-kernel of an AF $\mathcal{F} = (A, R)$ is $\mathcal{F}^{k(st)} = (A, R \backslash \{(a,b) \mid a \neq b, (a,a) \in R\})$.

Theorem 4. *For any AFs $\mathcal{F}$ and $\mathcal{G}$,*

$$\mathcal{F}^{k(st)} = \mathcal{G}^{k(st)} \Leftrightarrow \mathcal{F} \equiv^{st}_{\preceq^N_S} \mathcal{G}.$$

Proof. We only have to show that $\mathcal{F} \equiv^{st}_{\preceq^N_S} \mathcal{G} \Rightarrow \mathcal{F}^{k(st)} = \mathcal{G}^{k(st)}$ holds since $\mathcal{F}^{k(st)} = \mathcal{G}^{k(st)} \Rightarrow \mathcal{F} \equiv^{st}_{\preceq} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{st}_{\preceq^N} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{st}_{\preceq^N_S} \mathcal{G}$ is given by Theorem 1 and Prop. 3. We will prove this implication by contraposition.

Suppose $\mathcal{F}^{k(st)} \neq \mathcal{G}^{k(st)}$. $\mathbf{1}^{st}$ **case**: Consider $A(\mathcal{F}^{k(st)}) \neq A(\mathcal{G}^{k(st)})$. Consequently $A(\mathcal{F}) \neq A(\mathcal{G})$ and w.l.o.g. there exists an argument $a \in A(\mathcal{F}) \backslash A(\mathcal{G})$. Let c be a new argument, i.e. $c \notin A(\mathcal{F} \cup \mathcal{G})$ and $B = A(\mathcal{F} \cup \mathcal{G}) \backslash \{a\}$. We define

$$\mathcal{H} = (B \cup \{c\}, \{(c, c') \,|c' \in B\}).$$

If a is contained in some $E \in \mathcal{E}_{st}(\mathcal{F} \cup \mathcal{H})$, then $E \notin \mathcal{E}_{st}(\mathcal{G} \cup \mathcal{H})$ follows since $a \notin A(\mathcal{G} \cup \mathcal{H})$ was supposed. If not, consider $\mathcal{H}' = \mathcal{H} \cup (\{a\}, \varnothing)$. Then, $E = \{a, c\} \in \mathcal{E}_{st}(\mathcal{G} \cup \mathcal{H}')$ and $E \notin \mathcal{E}_{st}(\mathcal{F} \cup \mathcal{H}')$ since $\mathcal{F} \cup \mathcal{H}' = \mathcal{F} \cup \mathcal{H}$ holds. By definition of strong expansion equivalence $\mathcal{F} \not\equiv^{st}_{\preceq^N_S} \mathcal{G}$ follows.

$\mathbf{2}^{nd}$ **case**: We now consider $A(\mathcal{F}^{k(st)}) = A(\mathcal{G}^{k(st)})$ $(= A(\mathcal{F}) = A(\mathcal{G}))$ and $R(\mathcal{F}^{k(st)}) \neq R(\mathcal{G}^{k(st)})$. W.l.o.g. we may assume the existence of $a, b \in A(\mathcal{F})$, s.t. $(a,b) \in R(\mathcal{F}^{k(st)}) \backslash R(\mathcal{G}^{k(st)})$. Let c be a fresh argument. We define

$$\mathcal{I} = (A(\mathcal{F}) \cup \{c\}, \{(c, c') \,|c' \in A(\mathcal{F}) \backslash \{a, b\}\}).$$

Case **2.1:** Let $a = b$, therefore $(a,a) \in R(\mathcal{F}^{k(st)}) \backslash R(\mathcal{G}^{k(st)})$ and consequently $(a,a) \in R(\mathcal{F}) \backslash R(\mathcal{G})$ by the definition of the stable kernel. Applying splitting results $\mathcal{E}_{st}(\mathcal{G} \cup \mathcal{I}) = \{\{a, c\}\}$ and $\mathcal{E}_{st}(\mathcal{F} \cup \mathcal{I}) = \varnothing$ follow and hence, $\mathcal{F} \not\equiv^{st}_{\preceq^N_S} \mathcal{G}$ is shown. Thus from now on we assume that $R(\mathcal{F}^{k(st)})$ and $R(\mathcal{G}^{k(st)})$ contain the same self-loops. Case **2.2:** Let $a \neq b$. Since $(a,b) \in R(\mathcal{F}^{k(st)}) \backslash R(\mathcal{G}^{k(st)})$, it follows that $(a,b) \in R(\mathcal{F})$, $(a,a) \notin R(\mathcal{F})$, consequently $(a,a) \notin R(\mathcal{G})$ and $(a,b) \notin R(\mathcal{G})$. For all combinations w.r.t. the presence or absence of (b,a) and (b,b) in $\mathcal{F}$ and $\mathcal{G}$ we state: First, $\{a,c\} \in \mathcal{E}_{st}(\mathcal{F} \cup \mathcal{I})$ since $\{a,c\} \in cf(\mathcal{F} \cup \mathcal{I})$ and $R^+_{\mathcal{F} \cup \mathcal{I}}(\{a,c\}) = A(\mathcal{F} \cup \mathcal{I})$ and second, $\{a,c\} \notin \mathcal{E}_{st}(\mathcal{G} \cup \mathcal{I})$ since $R^+_{\mathcal{G} \cup \mathcal{I}}(\{a,c\}) = A(\mathcal{G} \cup \mathcal{I}) \backslash \{b\}$. Hence, $\mathcal{F} \not\equiv^{st}_{\preceq^N_S} \mathcal{G}$ follows. □

4.2. Strong Expansion Equivalence for Semi-Stable Semantics

Semi-stable semantics is, as the name suggests, very close to stable semantics. It can be shown that any stable extension is semi-stable and furthermore, if there is at least one stable extension then the set of stable and semi-stable extensions coincide [19]. In spite of these similarities the characterizing kernels of the correspondent strong equivalence notions differ. Oikarinen and Woltran showed that the equality of the more restrictive *ad*-kernel of two AFs adequately determine strong equivalence w.r.t. semi-stable semantics (compare Theorems 1 and 3). Apart from this, the following theorem states a similar result to Theorem 4, namely that the equality of the *ad*-kernels of two AFs is even necessarily for their strong expansion equivalence w.r.t. semi-stable semantics. Remember that the *ad*-kernel of an AF $\mathcal{F} = (A, R)$ is $\mathcal{F}^{k(ad)} = (A, R \backslash \{(a,b) \mid a \neq b, (a,a) \in R, \{(b,a),(b,b) \cap R \neq \varnothing\}\})$.

Theorem 5. *For any AFs $\mathcal{F}$ and $\mathcal{G}$,*

$$\mathcal{F}^{k(ad)} = \mathcal{G}^{k(ad)} \Leftrightarrow \mathcal{F} \equiv^{ss}_{\preceq^N_S} \mathcal{G}.$$

Proof. Observe that Theorems 1, 3 and Prop. 3 guarantee $\mathcal{F}^{k(ad)} = \mathcal{G}^{k(ad)} \Rightarrow \mathcal{F} \equiv^{ss}_{\preceq^N_S} \mathcal{G}$. Hence, it suffices to show that $\mathcal{F}^{k(ad)} = \mathcal{G}^{k(ad)}$ is implied by $\mathcal{F} \equiv^{ss}_{\preceq^N_S} \mathcal{G}$. We show the contrapositive.

Assume $\mathcal{F}^{k(ad)} \neq \mathcal{G}^{k(ad)}$. We have to show $\mathcal{F} \not\equiv^{ss}_{\preceq^N_S} \mathcal{G}$ which we do by case analysis. In almost all cases (except for the case 2.2.4) we even prove that, given the assumption, $\mathcal{F} \not\equiv^{\sigma}_{\preceq^N_S} \mathcal{G}$ for every $\sigma \in \{ss, ad, pr, id\}$. This can be shown without extra effort.[5] **1st case**: Assume $A(\mathcal{F}^{k(ad)}) \neq A(\mathcal{G}^{k(ad)})$. Hence, $A(\mathcal{F}) \neq A(\mathcal{G})$ is implied and w.l.o.g. there exists an argument $a \in A(\mathcal{F}) \backslash A(\mathcal{G})$. Let c be a fresh argument, i.e. $c \notin A(\mathcal{F} \cup \mathcal{G})$ and $B = A(\mathcal{F} \cup \mathcal{G}) \backslash \{a\}$. We define

$$\mathcal{H} = (B \cup \{c\}, \{(c,c') \,|c' \in B\}).$$

Given $\sigma \in \{ss, ad, pr, id\}$. If a is contained in some $E \in \mathcal{E}_\sigma(\mathcal{F} \cup \mathcal{H})$, then $E \notin \mathcal{E}_\sigma(\mathcal{G} \cup \mathcal{H})$ follows since $a \notin A(\mathcal{G} \cup \mathcal{H})$ was supposed. If not, consider $\mathcal{H}' = \mathcal{H} \cup (\{a\}, \varnothing)$. Applying splitting results it follows that $E = \{a, c\}$ is the unique preferred extension of $\mathcal{G} \cup \mathcal{H}'$. Consequently, E is admissible in $\mathcal{G} \cup \mathcal{H}'$ and the unique semi-stable and ideal extension of $\mathcal{G} \cup \mathcal{H}'$ (Prop. 1). On the other hand, $E \notin \mathcal{E}_\sigma(\mathcal{F} \cup \mathcal{H}')$ since $\mathcal{F} \cup \mathcal{H}' = \mathcal{F} \cup \mathcal{H}$ holds. This means, $\mathcal{F} \not\equiv^{\sigma}_{\preceq^N_S} \mathcal{G}$ for $\sigma \in \{ss, ad, pr, id\}$ is shown since $\mathcal{F}$ and $\mathcal{G}$ combined with $\mathcal{H}$ or $\mathcal{H}'$ are strong expansions of $\mathcal{F}$ and $\mathcal{G}$.

2nd case: Consider now $R(\mathcal{F}^{k(ad)}) \neq R(\mathcal{G}^{k(ad)})$ and $A(\mathcal{F}^{k(ad)}) = A(\mathcal{G}^{k(ad)})$. Note that $A(\mathcal{F}) = A(\mathcal{G}))$ is implied and furthermore, w.l.o.g. we may assume the existence of arguments $a, b \in A(\mathcal{F})$, s.t. $(a,b) \in R(\mathcal{F}^{k(ad)}) \backslash R(\mathcal{G}^{k(ad)})$. Let c be a fresh argument, i.e. $c \notin A(\mathcal{F})$. Furthermore we define

$$\mathcal{I} = (A(\mathcal{F}) \cup \{c\}, \{(c,c') \,|c' \in A(\mathcal{F}) \backslash \{a,b\}\}).$$

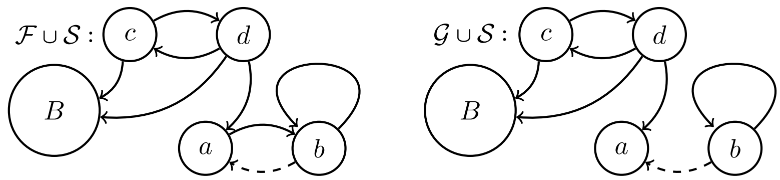
[5]We will use these results in Section 5.2 to prove that the admissible kernel adequately describes normal expansion equivalence w.r.t. admissible, preferred and ideal semantics.

Case **2.1:** Assume $a = b$. Therefore $(a,a) \in R(\mathcal{F}^{k(ad)}) \backslash R(\mathcal{G}^{k(ad)})$ and consequently $(a,a) \in R(\mathcal{F}) \backslash R(\mathcal{G})$ by definition of the admissible kernel. It can be checked (splitting results) that $\mathcal{E}_{pr}(\mathcal{G} \cup \mathcal{I}) = \{\{a,c\}\}$ and $\mathcal{E}_{pr}(\mathcal{F} \cup \mathcal{I}) = \{\{c\}\}$. Hence, $\mathcal{F} \cup \mathcal{I} \not\equiv^{\sigma} \mathcal{G} \cup \mathcal{I}$ for $\sigma \in \{ss, ad, pr, id\}$ can be obtained and therefore, $\mathcal{F} \not\equiv^{\sigma}_{\preceq^N_S} \mathcal{G}$ is shown. Thus from now on we assume that $R(\mathcal{F}^{k(ad)})$, $R(\mathcal{G}^{k(ad)})$, $R(\mathcal{F})$ and $R(\mathcal{G})$ contain the same self-loops.

Case **2.2:** Let $a \neq b$. Since $(a,b) \in R(\mathcal{F}^{k(ad)}) \backslash R(\mathcal{G}^{k(ad)})$, it follows $(a,b) \in R(\mathcal{F})$. Now we have to distinguish four cases w.r.t. the presence or absence of the self-loops (a,a) and (b,b). Case **2.2.1:** Assume $(a,a),(b,b) \in R(\mathcal{F})$. This case is impossible because the definition of the admissible kernel enforce the deletion of (a,b) in $R(\mathcal{F}^{k(ad)})$. Case **2.2.2**: Consider $(a,a) \in R(\mathcal{F})$ and $(b,b) \notin R(\mathcal{F})$. We observe that $(b,a) \notin R(\mathcal{F})$ holds (compare admissible kernel). Hence, $\mathcal{E}_{pr}(\mathcal{F} \cup \mathcal{I}) = \{\{c\}\}$ (splitting results). For $\mathcal{G}$ three cases arise. First, $(a,b) \in R(\mathcal{G})$ and consequently $(b,a) \in R(\mathcal{G})$ because of the assumption $(a,b) \notin R(\mathcal{G}^{k(ad)})$. Second and third, $(a,b) \notin R(\mathcal{G})$ and (b,a) may or may not be in $R(\mathcal{G})$. Using splitting results it can be checked that $\mathcal{E}_{pr}(\mathcal{G} \cup \mathcal{I}) = \{\{b,c\}\}$ holds. Thus, $\mathcal{F} \not\equiv^{\sigma}_{\preceq^N_S} \mathcal{G}$ for $\sigma \in \{ss, ad, pr, id\}$. Case **2.2.3**: Let $(a,a),(b,b) \notin R(\mathcal{F})$. We deduce $(a,b) \notin \mathcal{G}$ since $(a,a) \notin R(\mathcal{G})$ and $(a,b) \notin R(\mathcal{G}^{k(ad)})$ was assumed. We have to distinguish four sub-cases w.r.t. the presence or absence of (b,a). Suppose $(b,a) \notin R(\mathcal{F})$. Hence, $\mathcal{E}_{pr}(\mathcal{F} \cup \mathcal{I}) = \{\{a,c\}\}$. If $(b,a) \notin R(\mathcal{G})$, $\mathcal{E}_{pr}(\mathcal{G} \cup \mathcal{I}) = \{\{a,b,c\}\}$. If not, $\mathcal{E}_{pr}(\mathcal{G} \cup \mathcal{I}) = \{\{b,c\}\}$. In both cases $\mathcal{E}_{pr}(\mathcal{F} \cup \mathcal{I}) \neq \mathcal{E}_{pr}(\mathcal{G} \cup \mathcal{I})$ holds. Consider now $(b,a) \in R(\mathcal{F})$. It can be checked that $\mathcal{E}_{pr}(\mathcal{F} \cup \mathcal{I}) = \{\{a,c\},\{b,c\}\}$. Note that these sets are stable and therefore semi-stable extensions too. Furthermore, $\{c\}$ is admissible in $\mathcal{F} \cup \mathcal{I}$ and equals $\{a,c\} \cap \{b,c\}$. This means, $\mathcal{E}_{id}(\mathcal{F} \cup \mathcal{I}) = \{\{c\}\}$. Again, if $(b,a) \notin R(\mathcal{G})$, $\mathcal{E}_{pr}(\mathcal{G} \cup \mathcal{I}) = \{\{a,b,c\}\}$. If not, $\mathcal{E}_{pr}(\mathcal{G} \cup \mathcal{I}) = \{\{b,c\}\}$. Hence, in all cases $\mathcal{F} \cup \mathcal{I} \not\equiv^{\sigma} \mathcal{G} \cup \mathcal{I}$ for $\sigma \in \{ss, ad, pr, id\}$. Thus, $\mathcal{F} \not\equiv^{\sigma}_{\preceq^N_S} \mathcal{G}$ for $\sigma \in \{ss, ad, pr, id\}$ is shown. Case **2.2.4**: Consider $(a,a) \notin R(\mathcal{F})$ and $(b,b) \in R(\mathcal{F})$. As described at the very beginning of the proof this sub-case is the decisive point where semi-stable and admissible, preferred or ideal semantics behave different. We will only show that $\mathcal{F}$ and $\mathcal{G}$ are not strong expansion equivalent w.r.t. semi-stable semantics. In contrast to the other cases the AF $\mathcal{I}$ does not do the trick, i.e. $\mathcal{F} \cup \mathcal{I}$ and $\mathcal{G} \cup \mathcal{I}$ do not necessarily possess different semi-stable extensions. We therefore introduce a more sophisticated AF, namely

$$\mathcal{S} = (A(\mathcal{F}) \cup \{c,d\}, \{(e,f) \mid e \in \{c,d\} \wedge f \in A(\mathcal{F}) \backslash \{a,b\}\} \cup \{(c,d),(d,a),(d,c)\}).$$

The following figure illustrates $\mathcal{F} \cup \mathcal{S}$ and $\mathcal{G} \cup \mathcal{S}$. Note that $(a,b) \notin R(\mathcal{G})$ is implied since $(a,b) \notin R(\mathcal{G}^{k(ad)})$ and $(a,a) \notin R(\mathcal{G})$ is assumed. Remember that we already observed that in this case (b,a) may or not be in $R(\mathcal{F})$ or $R(\mathcal{G})$. The dashed arrows reflect this situation. The capital letter B is an abbreviation for the arguments in $A(\mathcal{F}) \backslash \{a,b\}$. Furthermore we left out possible attacks between B and $\{a,b\}$ since they are not important as we will see.

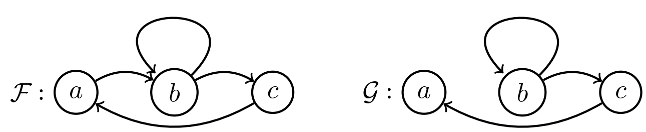


First notice that $\mathcal{E}_{ad}(\mathcal{F} \cup \mathcal{S}) = \mathcal{E}_{ad}(\mathcal{G} \cup \mathcal{S}) = \{\emptyset, \{a, c\}, \{c\}, \{d\}\}$. Remember that semi-stable extensions are admissible too. It turns out that $\{d\} \in \mathcal{E}_{ss}(\mathcal{G} \cup \mathcal{S})$ and $\{d\} \notin \mathcal{E}_{ss}(\mathcal{F} \cup \mathcal{S})$ holds. This can be seen as follows: In both AFs the ranges of $\{d\}$ are identical, i.e. $R^+_{\mathcal{F}\cup\mathcal{S}}(\{d\}) = R^+_{\mathcal{G}\cup\mathcal{S}}(\{d\}) = A(\mathcal{F} \cup \mathcal{S})\backslash\{b\}$. Since $R^+_{\mathcal{F}\cup\mathcal{S}}(\{a, c\}) = A(\mathcal{F} \cup \mathcal{S})$ we deduce $\{d\} \notin \mathcal{E}_{ss}(\mathcal{F} \cup \mathcal{S})$ by definition of the semi-stable semantics. On the other hand, for any set $E \in \mathcal{E}_{ad}(\mathcal{G} \cup \mathcal{S})$, $b \notin R^+_{\mathcal{G}\cup\mathcal{S}}(E)$ because $b \notin E$ and $(E, b) \notin R(\mathcal{G} \cup \mathcal{S})$. Hence, $R^+_{\mathcal{G}\cup\mathcal{S}}(\{d\}) \not\subseteq R^+_{\mathcal{G}\cup\mathcal{S}}(E)$. Consequently, $\{d\} \in \mathcal{E}_{ss}(\mathcal{G} \cup \mathcal{S})$ is shown and thus $\mathcal{F} \not\equiv^{ss}_{\preceq^N_S} \mathcal{G}$. □

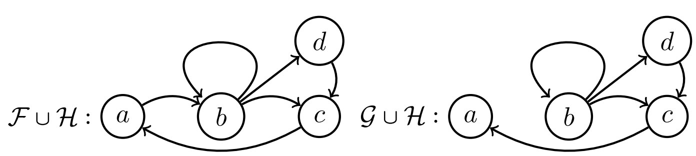
4.3. Strong Expansion Equivalence for Admissible, Preferred and Ideal Semantics

A special feature of strong expansions is that a former attack between old arguments will never become a counterattack to an added attack. In this sense, former attacks do not play a role w.r.t. being a potential defender of an added argument. Hence, in contrast to arbitrary expansions where such attacks might be relevant we may delete them without changing the behavior w.r.t. further evaluations. In the las two subsections we proved that in case of stable and semi-stable semantics there are no further redundant attacks if we consider strong expansion equivalence. In case of admissible, preferred and ideal semantics the situation becomes different. Consider the following example.

Example 6. The AFs $\mathcal{F}$ and $\mathcal{G}$ are not strongly equivalent w.r.t. admissible, preferred and ideal semantics since their corresponding *ad*-kernels $\mathcal{F}^{k(ad)} (= \mathcal{F})$ and $\mathcal{G}^{k(ad)} (= \mathcal{G})$ are different.



One possible scenario which makes the predicted different behaviour explicit is the following where $\mathcal{H} = (\{b, c, d\}, \{(b, d), (d, c)\})$. Observe that $\{a, d\} = \mathcal{E}_{pr}(\mathcal{F} \cup \mathcal{H}) \neq \{\emptyset\} = \mathcal{E}_{pr}(\mathcal{G} \cup \mathcal{H})$.



Note that the already existing attack (a,b) in $\mathcal{F}$ becomes a *defending* attack of the newly added argument d in the augmented argumentation scenario $\mathcal{F}\cup\mathcal{H}$. This means, such attacks in fact play an important role w.r.t. to further evaluation in case of arbitrary expansions. It is the main result of this section showing that AFs like $\mathcal{F}$ and $\mathcal{G}$ are strong expansion equivalent w.r.t. admissible, preferred and ideal semantics. This means, in particular, the attack (a,b) in $\mathcal{F}$ is redundant w.r.t. strong expansions and their evaluations.[6]

Here is the first novel kernel definition, the so-called *admissible-*-kernel* which (as we shall see) adequately describes strong expansion equivalence w.r.t. admissible, preferred and ideal semantics.

Definition 7. Given a an AF $\mathcal{F}=(A,R)$. We define the *admissible-*-kernel* of $\mathcal{F}$ as $\mathcal{F}^{k^*(ad)}=(A,R^{k^*(ad)})$ where

$$R^{k^*(ad)} = R \setminus \{(a,b)\mid a\neq b, ((a,a)\in R\wedge\{(b,a),(b,b)\}\cap R\neq\varnothing)\vee ((b,b)\in R\wedge\forall c\ ((b,c)\in R\rightarrow\{(a,c),(c,a),(c,c),(c,b)\}\cap R\neq\varnothing))\}.$$

The newly introduced kernel "forgets" an attack (a,b) if

1. a is self-attacking and at least one of the attacks (b,a) or (b,b) exists or
2. b is self-defeating and furthermore, for all arguments c which are attacked by b at least one of the following conditions holds: i) a attacks c, ii) c attacks a, iii) c attacks c or iv) c attacks b.

The first condition is exactly the same as in case of the admissible kernel (compare Def. 5). The motivation for the second disjunct is the following: At first observe that b cannot be an element of any conflict-free set. Thus, the attack (a,b) may only be relevant w.r.t. the defense of c since we are considering strong expansions. In the first three cases i), ii) and iii) this relevance becomes unimportant since $\{a,c\}$ does not even possess conflict-freeness. In case iv) the redundancy of (a,b) w.r.t. the defense of c is given by the fact that c already defends itself against b.

In the following we will prove that two AFs $\mathcal{F}$ and $\mathcal{G}$ possess the same admissible-*-kernel if and only if they are strong expansion equivalent w.r.t. admissible, preferred and ideal semantics. At first we will show that any AF $\mathcal{F}$ and its admissible-*-kernel possess the same extensions w.r.t. the aforementioned semantics.

Lemma 3. *For any AF $\mathcal{F}$ and $\sigma\in\{ad,pr,id\}$, $\mathcal{F}\equiv^{\sigma}\mathcal{F}^{k^*(ad)}$.*

PROOF. At first we show that $\mathcal{F}$ and $\mathcal{F}^{k^*(ad)}$ contain the same conflict-free sets, i.e. $S\in cf(\mathcal{F})$ iff $S\in cf(\mathcal{F}^{k^*(ad)})$. The if-direction is obvious because $R\left(\mathcal{F}^{k^*(ad)}\right)\subseteq R(\mathcal{F})$. It suffices to show that if $S\in cf(\mathcal{F}^{k^*(ad)})$, then $S\in cf(\mathcal{F})$. Assume not, i.e. there are at least two arguments $a,b\in S$, s.t. $(a,b)\in R(\mathcal{F})\setminus R\left(\mathcal{F}^{k^*(ad)}\right)$. Consequently, $(a,a)\in R(\mathcal{F})\vee(b,b)\in R(\mathcal{F})$ has to hold.

[6] We invite and encourage the reader to try to show that this assertion does not hold.

This contradicts the conflict-freeness of S in $\mathcal{F}^{k^*(ad)}$ because $\mathcal{F}^{k^*(ad)}$ and $\mathcal{F}$ share the same self-loops.

We now prove the result for $\sigma = ad$. We have to show that for each S conflict-free in $\mathcal{F}$ and $b \in S$, b is defended by S in $\mathcal{F}$ iff b is defended by S in $\mathcal{F}^{k^*(ad)}$. Hence, $\mathcal{F} \equiv^{ad} \mathcal{F}^{k^*(ad)}$ is implied. First, suppose b is defended by S in $\mathcal{F}$, i.e. for each $(a,b) \in R(\mathcal{F})$, $(S,\{a\}) \in R(\mathcal{F})$. Assume now b is not defended by S in $\mathcal{F}^{k^*(ad)}$, i.e. it exists $(a,b) \in R\left(\mathcal{F}^{k^*(ad)}\right)$, $(S,\{a\}) \notin R\left(\mathcal{F}^{k^*(ad)}\right)$. That means all counter-attacks $(c,a) \in R(\mathcal{F})$ have to be deleted. Since S is assumed to be conflict-free, $(c,c) \notin R(\mathcal{F})$ and hence, $(a,a) \in R(\mathcal{F})$ has to hold. If $c = b$, then $(a,b) \notin R\left(\mathcal{F}^{k^*(ad)}\right)$ because $(a,a) \in R(\mathcal{F})$ and $(b,a) \in R(\mathcal{F})$ was assumed. Let $c \neq b$. It follows $\{(b,c),(c,b),(b,b),(b,a)\} \cap R(\mathcal{F}) \neq \emptyset$. The first three attacks are impossible because conflict-freeness of S was assumed. Finally, if $(b,a) \in R(\mathcal{F})$, $(a,b) \notin R\left(\mathcal{F}^{k^*(ad)}\right)$ follows because $(a,a) \in R(\mathcal{F})$ was assumed.

Second, consider b is defended by S in $\mathcal{F}^{k^*(ad)}$ and b is not defended by S in $\mathcal{F}$, i.e. it exists $(a,b) \in R(\mathcal{F}) \setminus R\left(\mathcal{F}^{k^*(ad)}\right)$, $(S,\{a\}) \notin R(\mathcal{F})$. Since $(a,b) \notin R\left(\mathcal{F}^{k^*(ad)}\right)$, we deduce $(a,a) \in R(\mathcal{F})$ and $(b,b) \notin R(\mathcal{F})$ because conflict-freeness of S was assumed. Consequently, $(b,a) \in R(\mathcal{F})$ contradicting the assumption that b is not defended by S in $\mathcal{F}$. This concludes the proof for admissible semantics. Finally, applying Prop. 2, item 1 the claim is verified for preferred and ideal semantics. □

The following lemma states that, if two AFs $\mathcal{F}$ and $\mathcal{G}$ possess equal admissible-*-kernels, then the same holds for $\mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \cup \mathcal{H}$ where the latter AFs are strong expansions of the corresponding former ones.

Lemma 4. *If $\mathcal{F}^{k^*(ad)} = \mathcal{G}^{k^*(ad)}$, then $(\mathcal{F} \cup \mathcal{H})^{k^*(ad)} = (\mathcal{G} \cup \mathcal{H})^{k^*(ad)}$ for all AFs $\mathcal{H}$ which satisfy $\mathcal{F} \preceq_S^N \mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \preceq_S^N \mathcal{G} \cup \mathcal{H}$.*

Proof. First notice that the assumption $\mathcal{F}^{k^*(ad)} = \mathcal{G}^{k^*(ad)}$ implies $A(\mathcal{F}) = A(\mathcal{F}^{k^*(ad)}) = A(\mathcal{G}^{k^*(ad)}) = A(\mathcal{G})$. Given an AF $\mathcal{H}$, s.t. $\mathcal{F} \preceq_S^N \mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \preceq_S^N \mathcal{G} \cup \mathcal{H}$ is satisfied. Obviously, $(\mathcal{F} \cup \mathcal{H})^{k^*(ad)}$ and $(\mathcal{G} \cup \mathcal{H})^{k^*(ad)}$ share the same arguments. Hence, it suffices to show that $R\left((\mathcal{F} \cup \mathcal{H})^{k^*(ad)}\right) = R\left((\mathcal{G} \cup \mathcal{H})^{k^*(ad)}\right)$. Note that $\mathcal{F} = \mathcal{F} \cup \mathcal{H}$ if and only if $\mathcal{G} = \mathcal{G} \cup \mathcal{H}$. Hence, in case of equality we have nothing to show because $\mathcal{F}^{k^*(ad)} = \mathcal{G}^{k^*(ad)}$ guarantees $(\mathcal{F} \cup \mathcal{H})^{k^*(ad)} = (\mathcal{G} \cup \mathcal{H})^{k^*(ad)}$. This means, in the following we may assume that $\mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \cup \mathcal{H}$ are indeed strong expansions of $\mathcal{F}$ or $\mathcal{G}$. Consequently, $R(\mathcal{H}) \cap R(\mathcal{F}) = R(\mathcal{H}) \cap R(\mathcal{G}) = \emptyset$ can be assumed (compare Def. 2). Let $(a,b) \in R\left((\mathcal{F} \cup \mathcal{H})^{k^*(ad)}\right)$. We will show $(a,b) \in R\left((\mathcal{G} \cup \mathcal{H})^{k^*(ad)}\right)$ by proof by cases (containedness of a and b in $A(\mathcal{F})$ or $A(\mathcal{H}) \setminus A(\mathcal{F})$). Since $\mathcal{F}^{k^*(ad)} = \mathcal{G}^{k^*(ad)}$ is assumed, it suffices to consider $a \neq b$ because the sharing of the same self-loops of $(\mathcal{F} \cup \mathcal{H})^{k^*(ad)}$ and $(\mathcal{G} \cup \mathcal{H})^{k^*(ad)}$ is implied.

1st case: Let $a, b \in A(\mathcal{F})$. If $(a,b) \in R\left(\mathcal{F}^{k^*(ad)}\right)$, then $(a,b) \in R\left(\mathcal{G}^{k^*(ad)}\right)$ and $(a,b) \in R(\mathcal{G})$ follow. Furthermore $(a,b) \in R\left((\mathcal{G} \cup \mathcal{H})^{k^*(ad)}\right)$ since $\mathcal{G} \preceq_S^N$

$\mathcal{G} \cup \mathcal{H}$ was assumed, i.e. the AF $\mathcal{H}$ does not add relevant (w.r.t. the deletion of (a,b)) attacks. Assuming $(a,b) \notin R\left(\mathcal{F}^{k^*(ad)}\right)$ contradicts $(a,b) \in R\left((\mathcal{F} \cup \mathcal{H})^{k^*(ad)}\right)$ because the reason to remove an attack from $\mathcal{F} \cup \mathcal{H}$ remains untouched. **2**nd **case**: Let $a, b \in A(\mathcal{H}) \backslash A(\mathcal{F})$. Assume $(a,b) \notin R\left((\mathcal{G} \cup \mathcal{H})^{k^*(ad)}\right)$. Hence, several reasons for removing have to be considered. The first possibility is $(a,a) \in R(\mathcal{H}) \wedge \{(b,b),(b,a)\} \cap R(\mathcal{H}) \neq \varnothing$ holds. This implies $(a,b) \notin R\left((\mathcal{F} \cup \mathcal{H})^{k^*(ad)}\right)$. The second one is $(b,b) \in R(\mathcal{H}) \wedge \forall c\, ((b,c) \in R(\mathcal{G} \cup \mathcal{H}) \rightarrow \{(a,c),(c,a),(c,c),(c,b)\} \cap R(\mathcal{G} \cup \mathcal{H}) \neq \varnothing)$ holds. If the re is no c in $A(\mathcal{G})$ which is attacked by b we conclude $(a,b) \notin R\left((\mathcal{F} \cup \mathcal{H})^{k^*(ad)}\right)$ contradicting the assumption. So, consider $c \in A(\mathcal{G})$ and $(b,c) \in R(\mathcal{H})$. Consequently, $\{(a,c),(c,a),(c,c),(c,b)\} \cap R(\mathcal{G} \cup \mathcal{H}) \neq \varnothing$ has to hold. The attacks (c,a) and (c,b) are impossible since $\mathcal{G} \preceq_S^N \mathcal{G} \cup \mathcal{H}$ was assumed. If $(a,c) \in R(\mathcal{G} \cup \mathcal{H})$, then $(a,c) \in R(\mathcal{H})$ and consequently $(a,c) \in R(\mathcal{F} \cup \mathcal{H})$ has to hold. If $(c,c) \in R(\mathcal{G} \cup \mathcal{H})$, then $(c,c) \in R(\mathcal{G})$ and $(c,c) \in R(\mathcal{F})$ (since $\mathcal{F}^{k^*(ad)} = \mathcal{G}^{k^*(ad)}$ was assumed), therefore $(c,c) \in R(\mathcal{F} \cup \mathcal{H})$. In all cases, $(a,b) \notin R\left((\mathcal{F} \cup \mathcal{H})^{k^*(ad)}\right)$. **3**rd **case**: Let $a \in A(\mathcal{H}) \backslash A(\mathcal{F})$ and $b \in A(\mathcal{F})$. Assume $(a,b) \notin R\left((\mathcal{G} \cup \mathcal{H})^{k^*(ad)}\right)$. Again, several reasons for removing have to be considered. First consider $(a,a) \in R(\mathcal{H}) \wedge (b,b) \in R(\mathcal{G})$. We conclude $(b,b) \in R(\mathcal{F})$ because $\mathcal{F}^{k^*(ad)} = \mathcal{G}^{k^*(ad)}$ was assumed, thus $(a,a),(b,b) \in R(\mathcal{F} \cup \mathcal{H})$ holds which contradicts $(a,b) \in R\left((\mathcal{F} \cup \mathcal{H})^{k^*(ad)}\right)$. Note that $(a,a) \in R(\mathcal{H}) \wedge (b,a) \in R(\mathcal{G} \cup \mathcal{H})$ is impossible since $\mathcal{G} \preceq_S^N \mathcal{G} \cup \mathcal{H}$ was assumed. Consider now $(b,b) \in R(\mathcal{G}) \wedge \forall c\, ((b,c) \in R(\mathcal{G} \cup \mathcal{H}) \rightarrow \{(a,c),(c,a),(c,c),(c,b)\} \cap R(\mathcal{G} \cup \mathcal{H}) \neq \varnothing)$. We observe $(b,b) \in R(\mathcal{F})$. Since $(a,b) \in R\left((\mathcal{F} \cup \mathcal{H})^{k^*(ad)}\right)$ was assumed there exists an argument $c \in A(\mathcal{F})$, such that $(b,c) \in R(\mathcal{F}) \wedge \{(a,c),(c,a),(c,c),(c,b)\} \cap R(\mathcal{F} \cup \mathcal{H}) = \varnothing$ holds. Thus, $\{(a,c),(c,a),(c,c)\} \cap R(\mathcal{G} \cup \mathcal{H}) = \varnothing$ holds. Remember that we assumed $\mathcal{F}^{k^*(ad)} = \mathcal{G}^{k^*(ad)}$. If $(b,c) \notin R(\mathcal{G})$, then (b,c) has to be deleted in $R\left(\mathcal{F}^{k^*(ad)}\right)$. But this is impossible since we already concluded $(c,c) \notin R(\mathcal{F}) \wedge (c,b) \notin R(\mathcal{F})$. If $(b,c) \in R(\mathcal{G})$, then $(c,b) \in R(\mathcal{G})$ has to hold since we assumed $(a,b) \notin R\left((\mathcal{G} \cup \mathcal{H})^{k^*(ad)}\right)$. Hence, (b,c) has to be deleted in $\mathcal{G}^{k^*(ad)}$ because $(b,b) \in R(\mathcal{G})$ was supposed. This contradicts $(b,c) \in \mathcal{F}^{k^*(ad)}$ concluding the proof. **4**th **case**: Let $a \in A(\mathcal{F})$ and $b \in A(\mathcal{H}) \backslash A(\mathcal{F})$. Here we have nothing to show because the assumption $(a,b) \in R(\mathcal{F} \cup \mathcal{H})$ is impossible since $\mathcal{F} \preceq_S^N \mathcal{F} \cup \mathcal{H}$ was supposed. □

Now we are prepared to show that the syntactical equivalence of admissible-*-kernels characterizes strong expansion equivalence between two AFs $\mathcal{F}$ and $\mathcal{G}$ w.r.t. admissible, preferred and ideal semantics.

Theorem 6. *For any AFs $\mathcal{F}$, $\mathcal{G}$ and $\sigma \in \{ad, pr, id\}$:*

$$\mathcal{F}^{k^*(ad)} = \mathcal{G}^{k^*(ad)} \Leftrightarrow \mathcal{F} \equiv^{\sigma}_{\preceq_S^N} \mathcal{G}.$$

Proof. Let $\mathcal{F}^{k^*(ad)} = \mathcal{G}^{k^*(ad)}$. Given an AF $\mathcal{H}$, s.t. $\mathcal{F} \preceq_S^N \mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \preceq_S^N \mathcal{G} \cup \mathcal{H}$. It suffices to show that $E \in \mathcal{E}_{ad}(\mathcal{F} \cup \mathcal{H})$ implies $E \in \mathcal{E}_{ad}(\mathcal{G} \cup \mathcal{H})$. Suppose

$E \in \mathcal{E}_{ad}(\mathcal{F} \cup \mathcal{H})$. By Lemma 3, $E \in \mathcal{E}_{ad}\left((\mathcal{F} \cup \mathcal{H})^{k^*(ad)}\right)$ and applying Lemma 4, $E \in \mathcal{E}_{ad}\left((\mathcal{G} \cup \mathcal{H})^{k^*(ad)}\right)$. Finally, using Lemma 3, we derive $E \in \mathcal{E}_{ad}(\mathcal{G} \cup \mathcal{H})$ which concludes the if-direction for admissible semantics. Prop. 2 verifies the result for preferred and ideal semantics too, i.e. $\mathcal{F} \equiv^{pr}_{\leq^N_S} \mathcal{G}$ and $\mathcal{F} \equiv^{id}_{\leq^N_S} \mathcal{G}$.

We now show that $\mathcal{F}^{k^*(ad)} \neq \mathcal{G}^{k^*(ad)}$ implies $\mathcal{F} \not\equiv^{\sigma}_{\leq^N_S} \mathcal{G}$. $\mathbf{1}^{st}$ **case**: Assume $A\left(\mathcal{F}^{k^*(ad)}\right) \neq A\left(\mathcal{G}^{k^*(ad)}\right)$. Hence, w.l.o.g. exists an argument $a \in A(\mathcal{F}) \backslash A(\mathcal{G})$. We define $\mathcal{H} = ((A(\mathcal{F}) \cup A(\mathcal{G})) \backslash \{a\}, \varnothing)$. Consider the existence of a set E, s.t. $E \in \mathcal{E}_{\sigma}(\mathcal{F} \cup \mathcal{H})$ and $a \in E$. Consequently, $E \notin \mathcal{E}_{\sigma}(\mathcal{G} \cup \mathcal{H})$ holds. Assume now that for all extensions $E \in \mathcal{E}_{\sigma}(\mathcal{F} \cup \mathcal{H})$, $a \notin E$. We define $\mathcal{H}' = \mathcal{H} \cup (\{a\}, \varnothing)$. Hence, $\mathcal{F} \cup \mathcal{H} = \mathcal{F} \cup \mathcal{H}'$ and therefore, for all extensions $E \in \mathcal{E}_{\sigma}(\mathcal{F} \cup \mathcal{H}')$, $a \notin E$ holds. We observe $\{a\} \in \mathcal{E}_{ad}(\mathcal{G} \cup \mathcal{H}')$ and furthermore, for each $E \in \mathcal{E}_{pr}(\mathcal{G} \cup \mathcal{H}')$, $a \in E$ holds since a is unattacked in $\mathcal{G} \cup \mathcal{H}'$. This implies that a is contained in the ideal extensio n of $\mathcal{G} \cup \mathcal{H}'$. In all cases, $\mathcal{F} \not\equiv^{\sigma}_{\leq^N_S} \mathcal{G}$.

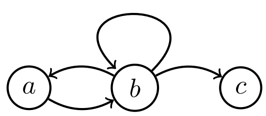
$\mathbf{2}^{nd}$ **case**: Consider now $R\left(\mathcal{F}^{k^*(ad)}\right) \neq R\left(\mathcal{G}^{k^*(ad)}\right)$ and $A\left(\mathcal{F}^{k^*(ad)}\right) = A\left(\mathcal{G}^{k^*(ad)}\right)$ $(= A(\mathcal{F}) = A(\mathcal{G}))$. Hence, w.l.o.g. there exists $a, b \in A(\mathcal{F})$, s.t. $(a,b) \in R\left(\mathcal{F}^{k^*(ad)}\right) \backslash R\left(\mathcal{G}^{k^*(ad)}\right)$. Let c be a new argument, i.e. $c \notin A(\mathcal{F})$. Furthermore we define

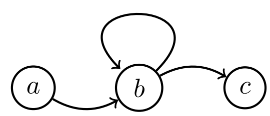
$$\mathcal{I} = (A(\mathcal{F}) \cup \{c\}, \{(c,c') \,|c' \in A(\mathcal{F}) \backslash \{a,b\}\}).$$

Case **2.1**: Assume $a = b$. This means $(a,a) \in R\left(\mathcal{F}^{k^*(ad)}\right) \backslash R\left(\mathcal{G}^{k^*(ad)}\right)$ and consequently $(a,a) \in R(\mathcal{F}) \backslash R(\mathcal{G})$ by the definition of the admissible-*-kernel. It can be checked (splitting results) that $\{a,c\}$ is an admissible and the unique preferred extension of $\mathcal{G} \cup \mathcal{I}$. Hence, it follows that $\{a,c\}$ has to be the unique ideal extension of $\mathcal{G} \cup \mathcal{I}$ (Prop. 1). On the other hand, we have $\{a,c\} \notin \mathcal{E}_{\sigma}(\mathcal{F} \cup \mathcal{I})$ $(\sigma \in \{ad, pr, id\})$ since $(a,a) \in R(\mathcal{F})$ was assumed. Thus from now on we assume that any self-loop is either contained in both $R\left(\mathcal{F}^{k^*(ad)}\right)$ and $R\left(\mathcal{G}^{k^*(ad)}\right)$ or in none of them.

Case **2.2**: Let $a \neq b$, i.e. $(a,b) \in R\left(\mathcal{F}^{k^*(ad)}\right) \backslash R\left(\mathcal{G}^{k^*(ad)}\right)$ and $(a,b) \in R(\mathcal{F})$. Now we have to distinguish four cases for the presence or absence of attack (a,a) and (b,b). Keep in mind that $R(\mathcal{F}), R(\mathcal{G}), R\left(\mathcal{F}^{k^*(ad)}\right)$ and $R\left(\mathcal{G}^{k^*(ad)}\right)$ contain the same self-loops. Case **2.2.1**: $(a,a),(b,b) \in R(\mathcal{F})$. This case is impossible because the definition of the admissible-*-kernel enforce the deletion of (a,b) in $R\left(\mathcal{F}^{k^*(ad)}\right)$. Case **2.2.2**: $(a,a),(b,b) \notin R(\mathcal{F})$. Note that $(a,b) \notin R(\mathcal{G})$ holds because a and b do not exhibit self-loops and $(a,b) \notin R\left(\mathcal{G}^{k^*(ad)}\right)$ was assumed. The attack (b,a) may or may not be an element of $R(\mathcal{F})$ or $R(\mathcal{G})$. The following results can be checked by using splitting results. If $(b,a) \notin R(\mathcal{F})$, then $\{\{a,c\}\} = \mathcal{E}_{\sigma}(\mathcal{F} \cup \mathcal{I})$ for any $\sigma \in \{pr, id\}$. If not, i.e. $(b,a) \in R(\mathcal{F})$, then $\{\{a,c\},\{b,c\}\} = \mathcal{E}_{pr}(\mathcal{F} \cup \mathcal{I})$ and $\{\{c\}\} = \mathcal{E}_{id}(\mathcal{F} \cup \mathcal{I})$. On the other hand, if $(b,a) \notin R(\mathcal{G})$, then $\{\{a,b,c\}\} = \mathcal{E}_{\sigma}(\mathcal{G} \cup \mathcal{I})$ holds for any $\sigma \in \{pr, id\}$. If not, i.e. $(b,a) \in R(\mathcal{G})$ it follows $\{\{b,c\}\} = \mathcal{E}_{\sigma}(\mathcal{G} \cup \mathcal{I})$ for any $\sigma \in \{pr, id\}$. Thus, in all possible combinations we obtain different preferred and ideal extensions.

Furthermore, different admissible extensions are implied (Prop. 2, item 1). This means, we have shown that for any $\sigma \in \{ad, pr, id\}$, $\mathcal{F} \not\equiv^{\sigma}_{\preceq^N_S} \mathcal{G}$ holds. Case **2.2.3**: $(a,a) \in R(\mathcal{F})$ and $(b,b) \notin R(\mathcal{F})$. First notice that $(b,a) \in R(\mathcal{F})$ cannot hold because $(a,a) \in R(\mathcal{F})$ would enforce the deletion of (a,b) in $R\left(\mathcal{F}^{k^*(ad)}\right)$ in contrast to the assumption. Using the standard construction we obtain $\{\{c\}\} = \mathcal{E}_\sigma(\mathcal{F} \cup \mathcal{I})$ for each $\sigma \in \{pr, id\}$. In the given self-loop constellation AF $\mathcal{G}$ may occur in three configuration w.r.t. the presence and absence of the attacks (a,b) and (b,a), namely: $(a,b),(b,a) \notin R(\mathcal{G})$ or $(a,b),(b,a) \in R(\mathcal{G})$ or $(a,b) \notin R(\mathcal{G})$ and $(b,a) \in R(\mathcal{G})$. Note that $(a,b) \in R(\mathcal{G})$ and $(b,a) \notin R(\mathcal{G})$ is impossible since $(a,b) \notin R\left(\mathcal{G}^{k^*(ad)}\right)$ was assumed. In all cases we obtain $\{\{b,c\}\} = \mathcal{E}_\sigma(\mathcal{G} \cup \mathcal{I})$ for each $\sigma \in \{pr, id\}$. By Prop. 2 we deduce $\mathcal{E}_{ad}(\mathcal{F} \cup \mathcal{I}) \neq \mathcal{E}_{ad}(\mathcal{G} \cup \mathcal{I})$. Altogether, we have shown that $\mathcal{F} \not\equiv^{\sigma}_{\preceq^N_S} \mathcal{G}$ for each $\sigma \in \{ad, pr, id\}$. Case **2.2.4**: $(a,a) \notin R(\mathcal{F})$ and $(b,b) \in R(\mathcal{F})$. Since $(a,b) \in R\left(\mathcal{F}^{k^*(ad)}\right)$ is assumed, we deduce the existence of an argument $c \in A(\mathcal{F})$, s.t. $(b,c) \in R(\mathcal{F}) \wedge \{(a,c),(c,a),(c,c),(c,b)\} \cap R(\mathcal{F}) = \emptyset$ (compare the definition of the admissible-*-kernel). The following figures show the remaining two possibilities for AF $\mathcal{F}$. Note that we omit possible other arguments than a, b and c. This means, the AFs $\mathcal{F}_1$ and $\mathcal{F}_2$ as well as the subsequent AFs $\mathcal{G}_i$ are only representatives illustrating the relevant parts (consult Section 2.3).

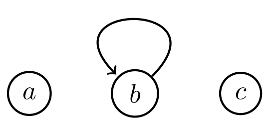
$\mathcal{F}_1 :$ 

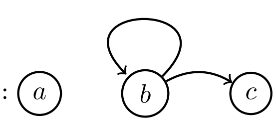
$\mathcal{F}_2 :$ 

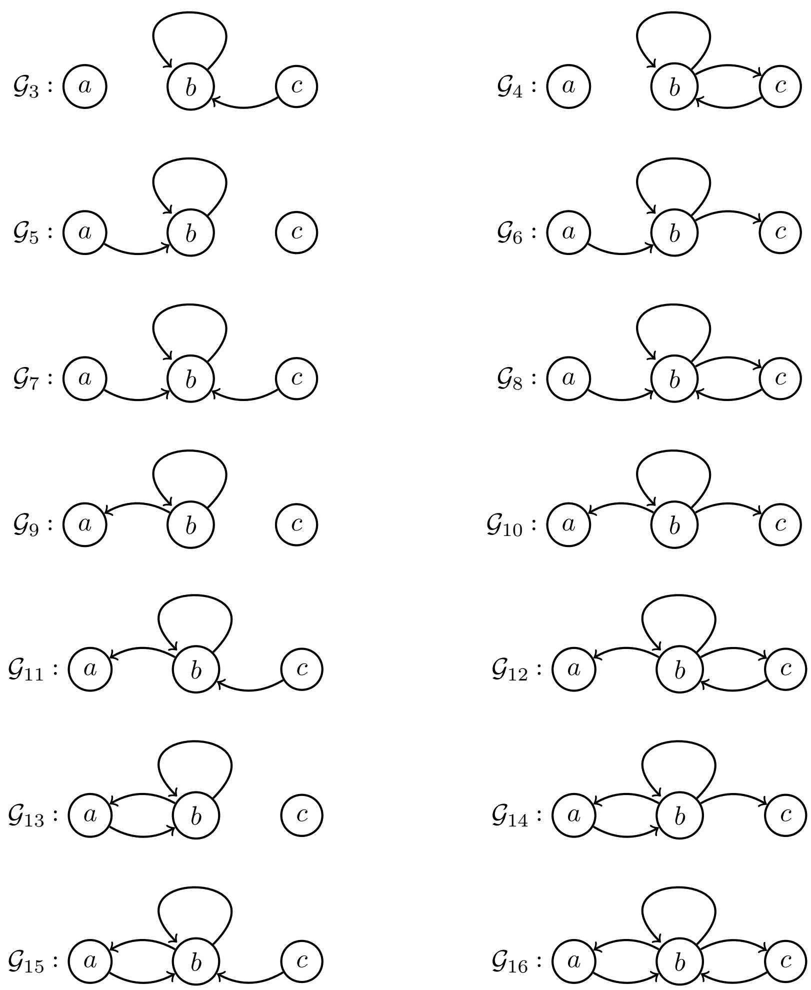
So far we know $(a,a),(c,c) \notin R(\mathcal{G})$ and $(b,b) \in R(\mathcal{G})$. This means there are $2^6 = 64$ possibilities for the presence and absence of $(a,b),(b,a),(b,c),(c,b),(a,c)$ and (c,a) in $R(\mathcal{G})$. Note that some of them are impossible since $(a,b) \notin R\left(\mathcal{G}^{k^*(ad)}\right)$ was assumed. At first we modify the standard construction in the following way (d is a fresh argument):

$$\mathcal{I}' = (A(\mathcal{F}) \cup \{d\}, \{(d,c') \,|\, c' \in A(\mathcal{F}) \backslash \{a,b,c\}\}).$$

The following extensions can be checked by applying splitting results (cf. Example 3). It can be easily seen that for each $\sigma \in \{pr, id\}$, $\mathcal{E}_\sigma(\mathcal{F}_1 \cup \mathcal{I}') = \mathcal{E}_\sigma(\mathcal{F}_2 \cup \mathcal{I}') = \{\{a,c,d\}\}$ holds. If $(a,c) \in R(\mathcal{G})$ or $(c,a) \in R(\mathcal{G})$, then for each $\sigma \in \{ad, pr, id\}$, $\{a,c,d\} \notin \mathcal{E}_\sigma(\mathcal{G} \cup \mathcal{H}')$ holds since $\{a,c,d\}$ is not conflict-free. Hence, w.l.o.g. we may assume $(a,c),(c,a) \notin R(\mathcal{G})$. Thus, $2^4 = 16$ possibilities w.r.t. the presence or absence of $(a,b),(b,a),(b,c)$ and (c,b) remain. For clarity, we will present all possibilities.

$\mathcal{G}_1 :$ 

$\mathcal{G}_2 :$ 



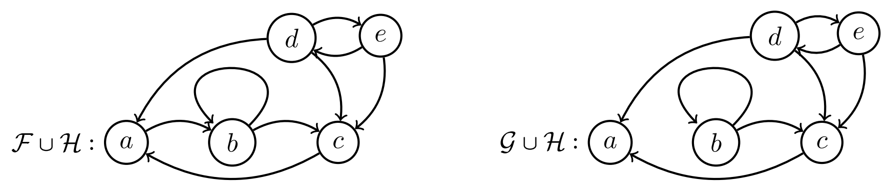
First note that $\mathcal{G}_6$ (= $\mathcal{F}_2$) and $\mathcal{G}_{14}$ (= $\mathcal{F}_1$) are impossible since $(a,b) \notin R\left(\mathcal{G}^{k^*(ad)}\right)$ was assumed. Furthermore, the cases $\mathcal{G}_2$, $\mathcal{G}_9$ and $\mathcal{G}_{10}$ can be checked by considering the union with AF $\mathcal{I}'$. For each $\sigma \in \{pr, id\}$, $\{\{a,d\}\} = \mathcal{E}_\sigma(\mathcal{G}_2 \cup \mathcal{I}')$, $\{\{c,d\}\} = \mathcal{E}_\sigma(\mathcal{G}_9 \cup \mathcal{I}')$ and $\{\{d\}\} = \mathcal{E}_\sigma(\mathcal{G}_{10} \cup \mathcal{I}')$. For all other cases we define a slightly different version of $\mathcal{I}'$, namely

$$\mathcal{I}'' = (A(\mathcal{F}) \cup \{d\}, \{(d,c') \,|c' \in A(\mathcal{F}) \backslash \{b,c\}\}) .$$

At first we have to check the extensions of $\mathcal{F}_1 \cup \mathcal{I}''$ and $\mathcal{F}_2 \cup \mathcal{I}''$. It turns out that for any $\sigma \in \{pr, id\}$, $\{\{d\}\} = \mathcal{E}_\sigma(\mathcal{F}_1 \cup \mathcal{I}'') = \mathcal{E}_\sigma(\mathcal{F}_2 \cup \mathcal{I}'')$ holds. On the other hand we have $\{\{c,d\}\} = \mathcal{E}_\sigma(\mathcal{G}_i \cup \mathcal{I}'')$ for each $\sigma \in \{pr, id\}$ and every $i \in \{1,3,4,5,7,8,11,12,13,15,16\}$. Remember that different preferred extensions imply different admissible extensions (Prop. 2, item 1). This means, finally, we have shown that for each $\sigma \in \{ad, pr, id\}$, $\mathcal{F} \not\equiv^\sigma_{\preceq^N_S} \mathcal{G}$ holds. □

Let us consider again Example 6 from the beginning of this section. According to Theorem 6 we have now formally proven that $\mathcal{F} \equiv^{\sigma}_{\preceq^N_S} \mathcal{G}$ for $\sigma \in \{ad, pr, id\}$ since both possess the same admissible-*-kernels, namely $\mathcal{F}^{k^*(ad)} = \mathcal{G}^{k^*(ad)} = \mathcal{G}$. In consideration of Theorem 5 the interested reader may ask for an example showing that $\mathcal{F}$ and $\mathcal{G}$ are not strong expansion equivalent w.r.t. semi-stable semantics. Here is a counter-example.

Example 7 (Example 6 cont.). Let $\mathcal{F}$ and $\mathcal{G}$ as defined in Example 6. We define $\mathcal{H} = (\{a, c, d, e\}, \{(d, a), (d, c), (d, e), (e, c), (e, d)\})$. The graph representation of $\mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \cup \mathcal{H}$ is as follows.



Observe that both possess the same admissible extensions, namely $\emptyset$, $\{a, e\}$, $\{d\}$ and $\{e\}$. Furthermore, in case of $\mathcal{F} \cup \mathcal{H}$ we have $\{a, c, d, e\} = R^+_{\mathcal{F}\cup\mathcal{H}}(\{d\}) \subset R^+_{\mathcal{F}\cup\mathcal{H}}(\{a, e\}) = \{a, b, c, d, e\}$. Due to the absence of the attack (a, b) in $\mathcal{G} \cup \mathcal{H}$ it can be easily seen that $\{d\}$ is a semi-stable extension in $\mathcal{G} \cup \mathcal{H}$. Hence, $\mathcal{F} \not\equiv^{ss}_{\preceq^N_S} \mathcal{G}$ follows.

4.4. Strong Expansion Equivalence for Grounded Semantics

Now we turn to the grounded semantics. Similarly to the case of admissible, preferred and ideal semantics we will see that strong expansion equivalence between two AFs is not sufficient for their strong equivalence w.r.t. grounded semantics. We therefore introduce a novel kernel, the so-called *grounded-*-kernel* which is defined as follows.

Definition 8. Given a an AF $\mathcal{F} = (A, R)$. We define the *grounded-*-kernel* of $\mathcal{F}$ as $\mathcal{F}^{k^*(gr)} = (A, R^{k^*(gr)})$ where

$$R^{k^*(gr)} = R \setminus \{(a, b) \mid a \neq b, ((b, b) \in R \wedge \{(a, a), (b, a)\} \cap R \neq \emptyset) \vee \\ ((b, b) \in R \wedge \forall c\ ((b, c) \in R \rightarrow \{(a, c), (c, a), (c, c)\} \cap R \neq \emptyset))\}.$$

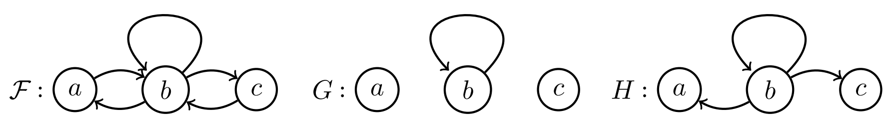
The newly introduced kernel "forgets" an attack (a, b) if

1. b is self-attacking and at least one of the attacks (a, a) or (b, a) exists or
2. b is self-defeating and furthermore, for all arguments c which are attacked by b at least one of the following conditions holds: i) a attacks c, ii) c attacks a or iii) c attacks c.

As explained in Section 4.3 a distinguishing feature of strong expansions in contrast to arbitrary expansions is that an old argument will never become a defender of a newly introduced and attacked argument. This means, there is more potential for unrelevant attacks which is reflected by the definition above.

The first disjunct captures attacks which are even redundant w.r.t. arbitrary expansions (compare gr-kernel, Def. 5). Similar to the definition of the admissible-*-kernel (Def. 7) the second disjunct allows the deletions of an attack (a,b) if b is self-attacking and for all c's which are attacked by b we have $\{a,c\}$ is conflicting encoded by i), ii) and iii). In these cases the potential defense of c by a becomes irrelevant since conflict-freeness is violated. In contrast to admissible, preferred and ideal semantics the fourth possibility, namely the presence of the attack (c,b), i.e. c defends itself against b does not justify a deletion of (a,b). This can be easily seen by considering the original definition of the grounded semantics introduced by Dung [2]. The grounded extension of an AF $\mathcal{F} = (A,R)$ is alternatively given as the least fix-point of the so-called *characteristic function* $\Gamma_{\mathcal{F}} : 2^A \to 2^A$, where $\Gamma_{\mathcal{F}}(S) = \{a \in A \mid a \text{ is defended by } S \text{ in } \mathcal{F}\}$. In case of finite AFs, this least fix-point can be achieved by applying iteratively $\Gamma_{\mathcal{F}}$ on the empty set. Furthermore, $\Gamma_{\mathcal{F}}$ can be shown to be monotonic [2]. This means, the fourth possibility is excluded because the defense of c against b by a may be essential for c being an element of the grounded extension, although c defends itself against b. The following example shows that we actually need a further kernel definition since the grounded extensions of an AF $\mathcal{F}$ and $\mathcal{F}^{k^*(ad)}$ are not necessarily the same.

Example 8. Observe that $\mathcal{F}^{k^*(ad)} = \mathcal{G}$ and $\mathcal{F}^{k^*(gr)} = \mathcal{H}$. Hence, $\mathcal{F} \equiv^{\sigma}_{\preceq^N_S} \mathcal{G}$ for any $\sigma \in \{ad, pr, id\}$ (Theorem 6). Furthermore, $\{\varnothing\} = \mathcal{E}_{gr}(\mathcal{F}) \neq \mathcal{E}_{gr}(\mathcal{G}) = \{\{a,c\}\}$ which proves $\mathcal{F} \not\equiv^{gr}_{\preceq^N_S} \mathcal{G}$. Note that the empty set is also the unique grounded extension of $\mathcal{H}$. We even claim that $\mathcal{F} \equiv^{gr}_{\preceq^N_S} \mathcal{H}$ which will be a consequence of Theorem 7, proved below.



Analogously to the subsection before we will proceed with two technical lemmata paving the way for the main theorem.

Lemma 5. *For any AF $\mathcal{F}$, $\mathcal{F} \equiv^{gr} \mathcal{F}^{k^*(gr)}$.*

Proof. It suffices to show that for all $i \geq 1$, $\Gamma^i_{\mathcal{F}}(\varnothing) = \Gamma^i_{\mathcal{F}^{k^*(gr)}}(\varnothing)$ holds. We will prove this by induction.

First, we show that the sets of unattacked arguments coincide, i.e. $\Gamma^1_{\mathcal{F}}(\varnothing) = \Gamma^1_{\mathcal{F}^{k^*(gr)}}(\varnothing)$. Furthermore, $\Gamma^1_{\mathcal{F}}(\varnothing) \subseteq \Gamma^1_{\mathcal{F}^{k^*(gr)}}(\varnothing)$ is obvious since $R\left(\mathcal{F}^{k^*(gr)}\right) \subseteq R(\mathcal{F})$ holds. Given $a \in \Gamma^1_{\mathcal{F}^{k^*(gr)}}(\varnothing)$, then $(a,a) \notin R\left(\mathcal{F}^{k^*(gr)}\right)$ and therefore $(a,a) \notin R(\mathcal{F})$. Assuming that a is attacked in $\mathcal{F}$, i.e. there is an argument b, s.t. $(b,a) \in R(\mathcal{F})$ yields to $(b,a) \in R\left(\mathcal{F}^{k^*(gr)}\right)$ (compare Def. 8). This contradicts the assumption that a is unattacked in $\mathcal{F}^{k^*(gr)}$.

Suppose now that for all $i < k$, $\Gamma^i_{\mathcal{F}}(\emptyset) = \Gamma^i_{\mathcal{F}^{k^*(gr)}}(\emptyset)$ holds. We will show that $\Gamma^k_{\mathcal{F}}(\emptyset) = \Gamma^k_{\mathcal{F}^{k^*(gr)}}(\emptyset)$ is implied. ($\subseteq$) Assume $a \in \Gamma^k_{\mathcal{F}}(\emptyset)$ and $a \notin \Gamma^k_{\mathcal{F}^{k^*(gr)}}(\emptyset)$. Hence there is an attack $(b,a) \in R\left(\mathcal{F}^{k^*(gr)}\right)$, s.t. b is not attacked by $\Gamma^{k-1}_{\mathcal{F}^{k^*(gr)}}(\emptyset)$ in $\mathcal{F}^{k^*(gr)}$. Since $a \in \Gamma^k_{\mathcal{F}}(\emptyset)$ was assumed it follows that there is at least one argument $c \in \Gamma^{k-1}_{\mathcal{F}}(\emptyset)$ ($\subseteq \Gamma^k_{\mathcal{F}}(\emptyset)$), s.t. $(c,b) \in R(\mathcal{F})$ holds (note that $b \notin \Gamma^{k-1}_{\mathcal{F}}(\emptyset)$ is implied). Consequently, all these attacks have to be deleted in $\mathcal{F}^{k^*(gr)}$. We have to consider several reasons for deletion. First notice that $(b,b) \in R(\mathcal{F})$ (thus $(b,b) \in R\left(\mathcal{F}^{k^*(gr)}\right)$) has to hold. Furthermore none of the attacks $(a,a),(a,c),(c,a) \in R(\mathcal{F})$ are possible since $(a,c) \in \Gamma^k_{\mathcal{F}}(\emptyset)$ has to be conflict-free. Hence, all arguments $c \in \Gamma^{k-1}_{\mathcal{F}}(\emptyset)$ with the property $(c,b) \in R(\mathcal{F})$ has to be counterattacked by b itself, i.e. $(b,c) \in R(\mathcal{F})$ (compare kernel-definition). Note that all these (b,c)'s survive in $R\left(\mathcal{F}^{k^*(gr)}\right)$ because $c \in \Gamma^{k-1}_{\mathcal{F}}(\emptyset)$ guarantees $(c,c) \notin R(\mathcal{F})$. By inductive hypothesis we get $c \in \Gamma^{k-1}_{\mathcal{F}^{k^*(gr)}}(\emptyset)$ and finally with $(b,c) \in R\left(\mathcal{F}^{k^*(gr)}\right)$ and the observation that all counter-attacks to b are deleted we contradict the admissibility of $\Gamma^{k-1}_{\mathcal{F}^{k^*(gr)}}(\emptyset)$ in $\mathcal{F}^{k^*(gr)}$. ($\supseteq$) Given $a \in \Gamma^k_{\mathcal{F}^{k^*(gr)}}(\emptyset)$, i.e. a is defended by $\Gamma^{k-1}_{\mathcal{F}^{k^*(gr)}}(\emptyset)$ in $\mathcal{F}^{k^*(gr)}$. Furthermore $(a,a) \notin R\left(\mathcal{F}^{k^*(gr)}\right)$ (thus $(a,a) \notin R(\mathcal{F})$) holds since $\Gamma^k_{\mathcal{F}^{k^*(gr)}}(\emptyset)$ is conflict-free in $\mathcal{F}^{k^*(gr)}$. This means, $(b,a) \in R\left(\mathcal{F}^{k^*(gr)}\right)$ iff $(b,a) \in R(\mathcal{F})$. Hence, using $\Gamma^{k-1}_{\mathcal{F}^{k^*(gr)}}(\emptyset) = \Gamma^{k-1}_{\mathcal{F}}(\emptyset)$ (inductive hypothesis) and the observation above we deduce that a is defended by $\Gamma^{k-1}_{\mathcal{F}}(\emptyset)$ in $\mathcal{F}$. Thus, $a \in \Gamma^k_{\mathcal{F}}(\emptyset)$. □

Lemma 6. *If $\mathcal{F}^{k^*(gr)} = \mathcal{G}^{k^*(gr)}$, then $(\mathcal{F} \cup \mathcal{H})^{k^*(gr)} = (\mathcal{G} \cup \mathcal{H})^{k^*(gr)}$ for all AFs $\mathcal{H}$ which satisfy $\mathcal{F} \preceq^N_S \mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \preceq^N_S \mathcal{G} \cup \mathcal{H}$.*

Proof. Assume $\mathcal{F}^{k^*(gr)} = \mathcal{G}^{k^*(gr)}$. Consequently, $A(\mathcal{F}) = A(\mathcal{F}^{k^*(gr)}) = A(\mathcal{G}^{k^*(gr)}) = A(\mathcal{G})$ holds. Consider now an AF $\mathcal{H}$ satisfying the specified properties (strong expansion or equality). Note that in case of equality there is nothing to show since $\mathcal{F} = \mathcal{F} \cup \mathcal{H}$ implies $\mathcal{G} = \mathcal{G} \cup \mathcal{H}$ and vice versa and hence, $(\mathcal{F} \cup \mathcal{H})^{k^*(gr)} = (\mathcal{G} \cup \mathcal{H})^{k^*(gr)}$ is implied. From now on we may suppose that $\mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \cup \mathcal{H}$ are indeed strong expansions of $\mathcal{F}$ or $\mathcal{G}$. Thus, $R(\mathcal{H}) \cap R(\mathcal{F}) = \emptyset$ and $R(\mathcal{H}) \cap R(\mathcal{G}) = \emptyset$ can be assumed (compare Def. 2). Let $(a,b) \in R\left((\mathcal{F} \cup \mathcal{H})^{k^*(gr)}\right)$, therefore $(a,b) \in R(\mathcal{F} \cup \mathcal{H})$. We will show $(a,b) \in R\left((\mathcal{G} \cup \mathcal{H})^{k^*(gr)}\right)$ by proof by cases (containedness of a and b in $A(\mathcal{F})$ or $A(\mathcal{H}) \backslash A(\mathcal{F})$). Again we suppose $a \neq b$ for all cases (containedness of self-loops is obvious).

1st **case**: Let $a,b \in A(\mathcal{F})$. If $(a,b) \in R\left(\mathcal{F}^{k^*(gr)}\right)$, then $(a,b) \in R\left(\mathcal{G}^{k^*(gr)}\right)$ and $(a,b) \in R(\mathcal{G})$ follow. Furthermore $(a,b) \in R\left((\mathcal{G} \cup \mathcal{H})^{k^*(gr)}\right)$ is implied because $\mathcal{G} \cup \mathcal{H}$ was assumed to be a strong expansion of $\mathcal{G}$ and so, no relevant attacks are added. The assumption $(a,b) \notin R\left(\mathcal{F}^{k^*(gr)}\right)$ contradicts $(a,b) \in R\left((\mathcal{F} \cup \mathcal{H})^{k^*(gr)}\right)$ because the reason to remove an attack remains untouched in $\mathcal{F} \cup \mathcal{H}$. **2**nd **case**: Let $a,b \in A(\mathcal{H}) \backslash A(\mathcal{F})$. Hence, $(a,b) \in R(\mathcal{G} \cup \mathcal{H})$ is implied. Assume now $(a,b) \notin R\left((\mathcal{G} \cup \mathcal{H})^{k^*(gr)}\right)$. This means, several reasons for removing have to be considered. Observe that $(b,b) \in R(\mathcal{H})$ has to hold. If (a,a) or (b,a) are contained in $R(\mathcal{H})$ we deduce $(a,b) \notin R\left((\mathcal{F} \cup \mathcal{H})^{k^*(gr)}\right)$ in contrast to the assumption. Assume now th at $\forall c\ ((b,c) \in R(\mathcal{G} \cup \mathcal{H}) \rightarrow \{(a,c),(c,a),(c,c)\} \cap R(\mathcal{G} \cup \mathcal{H}) \neq \varnothing)$ holds. If there is no c in $A(\mathcal{G})$ which is attacked by b we conclude $(a,b) \notin R\left((\mathcal{F} \cup \mathcal{H})^{k^*(gr)}\right)$. So, consider $c \in A(\mathcal{G})$ and $(b,c) \in R(\mathcal{H})$. We obtain $\{(a,c),(c,a),(c,c)\} \cap R(\mathcal{G} \cup \mathcal{H}) \neq \varnothing$. The attack $(c,a) \in R(\mathcal{G} \cup \mathcal{H})$ is impossible since $\mathcal{G} \preceq_S^N \mathcal{G} \cup \mathcal{H}$ was assumed. If $(a,c) \in R(\mathcal{G} \cup \mathcal{H})$, then $(a,c) \in R(\mathcal{H})$ and consequently $(a,c) \in R(\mathcal{F} \cup \mathcal{H})$ has to hold. If $(c,c) \in R(\mathcal{G} \cup \mathcal{H})$, then $(c,c) \in R(\mathcal{G})$ and $(c,c) \in R(\mathcal{F})$ (since $\mathcal{F}^{k^*(gr)} = \mathcal{G}^{k^*(gr)}$ was assumed), therefore $(c,c) \in R(\mathcal{F} \cup \mathcal{H})$. In all cases we get $(a,b) \notin R\left((\mathcal{F} \cup \mathcal{H})^{k^*(gr)}\right)$ in contrast to the assumption. **3rd case**: Let $a \in A(\mathcal{H}) \backslash A(\mathcal{F})$ and $b \in A(\mathcal{F})$. Hence, $(a,b) \in R(\mathcal{G} \cup \mathcal{H})$ is implied. Ass ume now $(a,b) \notin R\left((\mathcal{G} \cup \mathcal{H})^{k^*(gr)}\right)$. Again, several reasons for removing have to be considered. First, notice that $(b,b) \in R(\mathcal{G})$ (thus $(b,b) \in R(\mathcal{F})$) has to hold. If $(a,a) \in R(\mathcal{H})$ holds we deduce $(a,a),(b,b) \in R(\mathcal{F} \cup \mathcal{H})$ which contradicts $(a,b) \in R\left((\mathcal{F} \cup \mathcal{H})^{k^*(gr)}\right)$. Note that $(b,a) \in R(\mathcal{G} \cup \mathcal{H})$ is just impossible since $\mathcal{G} \preceq_S^N \mathcal{G} \cup \mathcal{H}$ was assumed. Assume now that $\forall c\ ((b,c) \in R(\mathcal{G} \cup \mathcal{H}) \rightarrow \{(a,c),(c,a),(c,c)\} \cap R(\mathcal{G} \cup \mathcal{H}) \neq \varnothing)$ holds. Since $(a,b) \in R\left((\mathcal{F} \cup \mathcal{H})^{k^*(gr)}\right)$ was assumed there exists an argument $c \in A(\mathcal{F})$, s.t. $(b,c) \in R(\mathcal{F}) \wedge \{(a,c),(c,a),(c,c)\} \cap R(\mathcal{F} \cup \mathcal{H}) = \varnothing$. We observe that $\{(a,c),(c,a),(c,c)\} \cap R(\mathcal{G} \cup \mathcal{H}) = \varnothing$ is implied and hence, if $(b,c) \in R(\mathcal{G})$, then $(a,b) \in R\left((\mathcal{G} \cup \mathcal{H})^{k^*(gr)}\right)$ follows in contrast to the assumption. Remember that $\mathcal{F}^{k^*(gr)} = \mathcal{G}^{k^*(gr)}$ has to hold. Hence, if $(b,c) \notin R(\mathcal{G})$, then (b,c) has to be deleted in $R\left(\mathcal{F}^{k^*(gr)}\right)$. This is impossible because $(c,c) \notin R(\mathcal{F} \cup \mathcal{H})$ (thus $(c,c) \notin R(\mathcal{F})$) is already shown. **4th case**: Let $a \in A(\mathcal{F})$ and $b \in A(\mathcal{H}) \backslash A(\mathcal{F})$. This case is impossible because $(a,b) \in R(\mathcal{F} \cup \mathcal{H})$ cannot hold if $\mathcal{F} \preceq_S^N \mathcal{F} \cup \mathcal{H}$ is fulfilled. □

With the help of the two lemmata above we will prove now that syntactical equivalence of grounded-*-kernels of two AFs characterizes their strong expansion equivalence w.r.t. grounded semantics.

Theorem 7. *For any AFs $\mathcal{F}$, $\mathcal{G}$:*

$$\mathcal{F}^{k^*(gr)} = \mathcal{G}^{k^*(gr)} \Leftrightarrow \mathcal{F} \equiv^{gr}_{\preceq_S^N} \mathcal{G}.$$

Proof. The if-direction, namely $\mathcal{F}^{k^*(gr)} = \mathcal{G}^{k^*(gr)} \Rightarrow \mathcal{F} \equiv^{gr}_{\preceq_S^N} \mathcal{G}$ follows by applying Lemmata 5 and 6 (similarly to Theorem 6). We will show the only-if-direction by proving the contrapositive, i.e. $\mathcal{F}^{k^*(gr)} \neq \mathcal{G}^{k^*(gr)} \Rightarrow \mathcal{F} \not\equiv^{gr}_{\preceq_S^N} \mathcal{G}$.

1st case: Assume $A\left(\mathcal{F}^{k^*(gr)}\right) \neq A\left(\mathcal{G}^{k^*(gr)}\right)$. Hence, w.l.o.g. exists an argument $a \in A(\mathcal{F}) \backslash A(\mathcal{G})$. We define $\mathcal{H} = ((A(\mathcal{F}) \cup A(\mathcal{G})) \backslash \{a\}, \varnothing)$. Let E be the unique grounded extension of $\mathcal{F} \cup \mathcal{H}$. If $a \in E$, $E \notin \mathcal{E}_{gr}(\mathcal{G} \cup \mathcal{H})$ follows. Consider now $a \notin E$. We define $\mathcal{H}' = \mathcal{H} \cup (\{a\}, \emptyset)$. Hence, $\mathcal{F} \cup \mathcal{H} = \mathcal{F} \cup \mathcal{H}'$ and therefore, E is the unique grounded extension of $\mathcal{F} \cup \mathcal{H}'$. Furthermore we observe that a is unattacked in $\mathcal{G} \cup \mathcal{H}'$ and so, a is contained in the unique grounded extension E' of $\mathcal{G} \cup \mathcal{H}'$. Hence, $\mathcal{F} \not\equiv^{gr}_{\preceq^N_S} \mathcal{G}$ follows.

2^{nd} case: Consider $R\left(\mathcal{F}^{k^*(gr)}\right) \neq R\left(\mathcal{G}^{k^*(gr)}\right)$ and $A\left(\mathcal{F}^{k^*(gr)}\right) = A\left(\mathcal{G}^{k^*(gr)}\right)$ $(= A(\mathcal{F}) = A(\mathcal{G}))$. Hence, w.l.o.g. there exists $a, b \in A(\mathcal{F})$, s.t. $(a,b) \in R\left(\mathcal{F}^{k^*(gr)}\right) \backslash R\left(\mathcal{G}^{k^*(gr)}\right)$. Let c be a new argument, i.e. $c \notin A(\mathcal{F})$. Furthermore we define

$$\mathcal{I} = (A(\mathcal{F}) \cup \{c\}, \{(c,c') \,|\, c' \in A(\mathcal{F}) \backslash \{a,b\}\}) .$$

Case **2.1**: Let $a = b$. This means $(a,a) \in R\left(\mathcal{F}^{k^*(gr)}\right) \backslash R\left(\mathcal{G}^{k^*(gr)}\right)$ and consequently $(a,a) \in R(\mathcal{F}) \backslash R(\mathcal{G})$ by the definition of the grounded-*-kernel. It is easy to see (splitting results) that $\{\{c\}\} = \mathcal{E}_{gr}(\mathcal{F} \cup \mathcal{I}) \neq \mathcal{E}_{gr}(\mathcal{G} \cup \mathcal{I}) = \{\{a,c\}\}$ holds. From now on we suppose that any self-loop is either contained in both $R\left(\mathcal{F}^{k^*(gr)}\right)$ and $R\left(\mathcal{G}^{k^*(gr)}\right)$ or in none of them.

Case **2.2**: Consider now $a \neq b$, i.e. $(a,b) \in R\left(\mathcal{F}^{k^*(gr)}\right) \backslash R\left(\mathcal{G}^{k^*(gr)}\right)$ and $(a,b) \in R(\mathcal{F})$. We have to distinguish four cases for the presence or absence of attack (a,a) and (b,b). Keep in mind that $R(\mathcal{F}), R(\mathcal{G}), R\left(\mathcal{F}^{k^*(gr)}\right)$ and $R\left(\mathcal{G}^{k^*(gr)}\right)$ contain the same self-loops. Case **2.2.1**: $(a,a),(b,b) \in R(\mathcal{F})$. This case is impossible because $(a,b) \in R\left(\mathcal{F}^{k^*(gr)}\right)$ cannot hold (grounded-*-kernel, Def. 8). Case **2.2.2**: $(a,a),(b,b) \notin R(\mathcal{F})$. Note that $(a,b) \notin R(\mathcal{G})$ holds because $(b,b) \notin R(\mathcal{G})$ and $(a,b) \notin R\left(\mathcal{G}^{k^*(gr)}\right)$ was assumed. The attack (b,a) may or may not be an element of $R(\mathcal{F})$ or $R(\mathcal{G})$. If $(b,a) \notin R(\mathcal{F})$, $\{\{a,c\}\} = \mathcal{E}_{gr}(\mathcal{F} \cup \mathcal{I})$ follows. If not, i.e. $(b,a) \in R(\mathcal{F})$, then $\{\{c\}\} = \mathcal{E}_{gr}(\mathcal{F} \cup \mathcal{I})$ holds. Furthermore, if $(b,a) \notin R(\mathcal{G})$ we deduce $\{\{a,b,c\}\} = \mathcal{E}_{gr}(\mathcal{G} \cup \mathcal{I})$ and if not, i.e. $(b,a) \in R(\mathcal{G})$ it follows $\{\{b,c\}\} = \mathcal{E}_{gr}(\mathcal{G} \cup \mathcal{I})$. Thus, in all possible combinations we obtain different grounded extensions, i.e. $\mathcal{F} \not\equiv^{gr}_{\preceq^N_S} \mathcal{G}$. **2.2.3**: $(a,a) \in R(\mathcal{F})$ and $(b,b) \notin R(\mathcal{F})$. Again, it is impossible that $(a,b) \in R(\mathcal{G})$ holds since $(b,b) \notin R(\mathcal{G})$ and $(a,b) \notin R\left(\mathcal{G}^{k^*(gr)}\right)$ was assumed. The attack (b,a) may or may not be an element of $R(\mathcal{F})$ and $R(\mathcal{G})$. Either way, $\{\{c\}\} = \mathcal{E}_{gr}(\mathcal{F} \cup \mathcal{I}) \neq \mathcal{E}_{gr}(\mathcal{G} \cup \mathcal{I}) = \{\{b,c\}\}$ follows. Hence, $\mathcal{F} \not\equiv^{gr}_{\preceq^N_S} \mathcal{G}$. **2.2.4**: $(a,a) \notin R(\mathcal{F})$ and $(b,b) \in R(\mathcal{F})$. Since $(a,b) \in R\left(\mathcal{F}^{k^*(gr)}\right)$ is assumed, we deduce $(b,a) \notin R(\mathcal{F})$ and furthermore the existence of an argument $c \in A(\mathcal{F}) : (b,c) \in R(\mathcal{F}) \wedge \{(a,c),(c,a),(c,c)\} \cap R(\mathcal{F}) = \emptyset$ (compare Def. 8). The following figures show the remaining two possibilities for AF $\mathcal{F}$. Note that we omit possible other arguments than a, b and c.

Up to now we know $(a,a),(c,c) \notin R(\mathcal{G})$ and $(b,b) \in R(\mathcal{G})$. Hence, there are $2^6 = 64$ possibilities for the presence and absence of $(a,b),(b,a),(b,c),(c,b),(a,c)$ and (c,a) in $R(\mathcal{G})$. We will show that some of them are impossible since $(a,b) \notin R\left(\mathcal{G}^{k^*(gr)}\right)$ was assumed. Again, we use the slightly different version of the standard construction $\mathcal{I}$, namely

$$\mathcal{I}' = (A(\mathcal{F}) \cup \{d\}, \{(d,c') \mid c' \in A(\mathcal{F}) \backslash \{a,b,c\}\}).$$

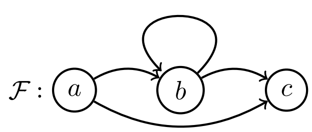
It can be checked that $\mathcal{E}_{gr}(\mathcal{F}_1 \cup \mathcal{I}') = \mathcal{E}_{gr}(\mathcal{F}_2 \cup \mathcal{I}') = \{\{a,c,d\}\}$. If $(a,c) \in R(\mathcal{G})$ or $(c,a) \in R(\mathcal{G})$, then $\{\{a,c,d\}\} \neq \mathcal{E}_{gr}(\mathcal{G} \cup \mathcal{I}')$ because a grounded extension has to be conflict-free. From now on we assume $(a,c),(c,a) \notin R(\mathcal{G})$. This means, $2^4 = 16$ possibilities w.r.t. the presence or absence of $(a,b),(b,a),(b,c)$ and (c,b) remain. These sixteen remaining possibilities are listed in Theorem 6 (see pp. 21-22). $\mathcal{G}_6$ (= $\mathcal{F}_2$) and $\mathcal{G}_8$ (= $\mathcal{F}_1$) are impossible since $(a,b) \notin R\left(\mathcal{G}^{k^*(gr)}\right)$ was assumed. The cases $\mathcal{G}_2$, $\mathcal{G}_4$, $\mathcal{G}_9$, $\mathcal{G}_{10}$, $\mathcal{G}_{12}$, $\mathcal{G}_{13}$, $\mathcal{G}_{14}$ and $\mathcal{G}_{16}$ can be checked by considering the union with AF $\mathcal{I}'$. For every $i \in \{2,4,9,10,12,13,14,16\}$, $\{\{a,c,d\}\} \neq \mathcal{E}_{gr}(\mathcal{G}_i \cup \mathcal{I}')$ holds. For all other cases we use

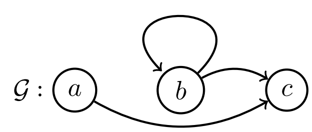
$$\mathcal{I}'' = (A(\mathcal{F}) \cup \{d\}, \{(d,c') \mid c' \in A(\mathcal{F}) \backslash \{b,c\}\}).$$

Combining $\mathcal{F}_1$ and $\mathcal{F}_2$ with $\mathcal{I}''$ we get $\{\{d\}\} = \mathcal{E}_{gr}(\mathcal{F}_1 \cup \mathcal{I}'') = \mathcal{E}_{gr}(\mathcal{F}_2 \cup \mathcal{I}'')$. Furthermore we have $\{\{c,d\}\} = \mathcal{E}_{gr}(\mathcal{G}_i \cup \mathcal{I}'')$ for every $i \in \{1,3,5,7,11,15\}$. Hence, $\mathcal{F} \not\equiv^{gr}_{\preceq^N_S} \mathcal{G}$ concluding the proof. □

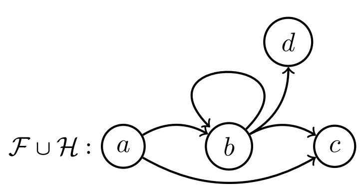
Finally, we will give a counter-example showing that strong expansion equivalence is not sufficient for strong equivalence w.r.t. grounded semantics as stated at the very beginning of this section.

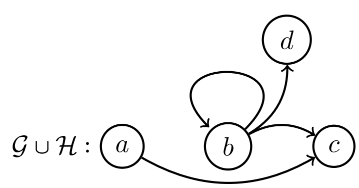
Example 9. The AFs $\mathcal{F}$ and $\mathcal{G}$ are strong expansion equivalent since they possess equal grounded-*-kernels, namely $\mathcal{F}^{k^*(gr)} = \mathcal{G}^{k^*(gr)} = \mathcal{G}$ (Theorem 7).





Furthermore, they are not strongly equivalent w.r.t. grounded semantics which can be demonstrated by the following expansions $\mathcal{F} \cup \mathcal{H}$ of $\mathcal{F}$ and $\mathcal{G} \cup \mathcal{H}$ of $\mathcal{G}$, where $\mathcal{H} = (\{b,d\},\{(b,d)\})$.





Using Dung's characteristic function we identify different grounded extensions for $\mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \cup \mathcal{H}$, namely $\{a,d\}$ resp. $\{a\}$.

4.5. Strong Expansion Equivalence for Complete Semantics

Finally, we consider the last novel kernel definition, the so-called *complete-*-kernel* which characterizes strong expansion equivalence w.r.t. complete semantics. Here is the formal definition.

Definition 9. Given an AF $\mathcal{F} = (A, R)$. We define the *complete-*-kernel* of $\mathcal{F}$ as $\mathcal{F}^{k^*(co)} = (A, R^{k^*(co)})$ where

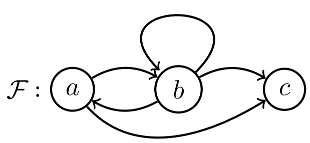
$$R^{k^*(co)} = R \setminus \{(a,b) \mid a \neq b, ((a,a),(b,b) \in R) \vee ((b,b) \in R \wedge (b,a) \notin R \wedge \forall c\, ((b,c) \in R \rightarrow \{(a,c),(c,a),(c,c)\} \cap R \neq \varnothing))\}.$$

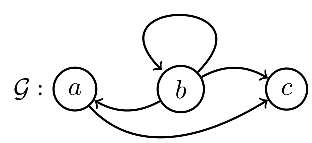
The newly introduced kernel "forgets" an attack (a,b) if

1. a and b are self-attacking or
2. b is self-defeating, b does not attack a and furthermore, for all arguments c which are attacked by b at least one of the following conditions holds: i) a attacks c, ii) c attacks a or iii) c attacks c.

The first disjunct describes attacks which are even redundant w.r.t. arbitrary expansions (compare *gr*-kernel, Def. 5). The additional part (second disjunct) of the complete-*-kernel is very similar to the grounded-*-kernel (Def. 8). The difference is that the deletion of an attack (a,b) requires the additional precondition that b does not attack a. This is due to the fact that the attack (a,b) may be crucial for the acceptance of the argument a if (b,a) is established. Roughly speaking, the argument a may justify its acceptance itself in contrast to grounded semantics where the reason for being a member of the unique grounded extension has to come from the outside, i.e. former accepted arguments have to defend a. Consider the following example.

Example 10. The AFs $\mathcal{F}$ and $\mathcal{G}$ $(= \mathcal{F}^{k^*(gr)})$ are strong expansion equivalent w.r.t. grounded semantics (Theorem 7). In particular, the deletion of (a,b) is irrelevant w.r.t. the grounded extensions of $\mathcal{F}$ and $\mathcal{G}$. Observe that in the case of complete semantics (a,b) is essential since $\{a\}$ is no longer complete in $\mathcal{G}$.





We proceed with some useful properties of the newly introduced kernel. The following lemma showes that any AF $\mathcal{F}$ and its complete-*-kernel possess the same complete extensions.

Lemma 7. *For any AF* $\mathcal{F}$*,* $\mathcal{F} \equiv^{co} \mathcal{F}^{k^*(co)}$.

Proof. The first step is to show that $\mathcal{F}$ and $\mathcal{F}^{k^*(co)}$ contain the same conflict-free sets, i.e. $S \in cf(\mathcal{F})$ iff $S \in cf(\mathcal{F}^{k^*(co)})$. The if-direction is obvious because

$R\left(\mathcal{F}^{k^*(co)}\right) \subseteq R(\mathcal{F})$ holds (complete-*-kernel). Assume now $S \in cf(\mathcal{F}^{k^*(co)})$ and $S \notin cf(\mathcal{F})$. Consequently, there are two arguments a and b in S with the property $(a,b) \in R(\mathcal{F})\backslash R\left(\mathcal{F}^{k^*(co)}\right)$. In any case, $(b,b) \in R(\mathcal{F})$ has to hold. The same applies to $R\left(\mathcal{F}^{k^*(co)}\right)$ which contradicts the assumption $S \in cf(\mathcal{F}^{k^*(ad)})$.

We now prove that $E \in \mathcal{E}_{co}(\mathcal{F})$ implies $E \in \mathcal{E}_{co}(\mathcal{F}^{k^*(co)})$. At first we will show that E is admissible in $\mathcal{F}^{k^*(co)}$. Assume $E \in \mathcal{E}_{co}(\mathcal{F})$ and E do not defend all its elements in $\mathcal{F}^{k^*(co)}$. This means, there is an argument $a \in E$ and an argument $b \notin E$ (conflict-freeness) s.t. $(b,a) \in R\left(\mathcal{F}^{k^*(co)}\right)$ and $(E,b) \notin R\left(\mathcal{F}^{k^*(co)}\right)$. Since $R\left(\mathcal{F}^{k^*(co)}\right) \subseteq R(\mathcal{F})$ and $\mathcal{E}_{co}(\mathcal{F}) \subseteq \mathcal{E}_{ad}(\mathcal{F})$ hold, we deduce the existence of an argument $c \in E$, s.t. $(c,b) \in R(\mathcal{F})\backslash R\left(\mathcal{F}^{k^*(co)}\right)$. There are two possibilities for the deletion of (c,b) in $R\left(\mathcal{F}^{k^*(co)}\right)$. First, $(c,c),(b,b) \in R(\mathcal{F})$ and second, $(b,b) \in R(\mathcal{F})$, $(b,c) \notin R(\mathcal{F})$ and at least $\{(a,c),(c,a),(a,a)\} \cap R(\mathcal{F}) \neq \varnothing$. Due to the conflict-freeness of E in $\mathcal{F}$ and the membership of a and c in E both options fail. Assume now $E \in \mathcal{E}_{co}(\mathcal{F})$ but E does not contain all defended elements in $\mathcal{F}^{k^*(co)}$. Hence, there is an argument $a \notin E$, s.t. for all arguments c with $(c,a) \in R\left(\mathcal{F}^{k^*(co)}\right), (E,c) \in R\left(\mathcal{F}^{k^*(co)}\right)$. Since E is assumed to be complete in $\mathcal{F}$ and $a \notin E$ we deduce the existence of an argument c with the property $(c,a) \in R(\mathcal{F})$ and $(E,c) \notin R(\mathcal{F})$. Combining both conclusions we get $(c,a) \in R(\mathcal{F})\backslash R\left(\mathcal{F}^{k^*(co)}\right)$. In any case, $(a,a) \in R(\mathcal{F})$ and thus, $(a,a) \in R\left(\mathcal{F}^{k^*(co)}\right)$. Since a is defended by E in $\mathcal{F}^{k^*(co)}$, $(E,a) \in R\left(\mathcal{F}^{k^*(co)}\right)$ has to hold. Finally, $(E,E) \in R\left(\mathcal{F}^{k^*(co)}\right)$ follows contradicting the conflict-freeness of E in $\mathcal{F}^{k^*(co)}$.

We now prove that $E \in \mathcal{E}_{co}(\mathcal{F}^{k^*(co)})$ implies $E \in \mathcal{E}_{co}(\mathcal{F})$. First of all, we show the admissibility of E in $\mathcal{F}$. Given $E \in \mathcal{E}_{co}(\mathcal{F}^{k^*(co)})$, we assume the existence of an argument $a \in E$ and an argument $b \notin E$ (conflict-freeness), s.t. $(b,a) \in R(\mathcal{F})$ and $(E,b) \notin R(\mathcal{F})$ holds. Due to the relations $R\left(\mathcal{F}^{k^*(co)}\right) \subseteq R(\mathcal{F})$ and $\mathcal{E}_{co}(\mathcal{F}^{k^*(co)}) \subseteq \mathcal{E}_{ad}(\mathcal{F}^{k^*(co)})$, $(b,a) \in R(\mathcal{F})\backslash R\left(\mathcal{F}^{k^*(co)}\right)$ follows. Consequently, $(a,a) \in R(\mathcal{F})$ has to hold contradicting the conflict-freeness of E in $\mathcal{F}$. Assume now that E do not contain all defended elements in $\mathcal{F}$, i.e. it exists an argument $a \notin E$, s.t. for all arguments c with the property $(c,a) \in R(\mathcal{F}), (E,c) \in R(\mathcal{F})$ holds. Since E is assumed to be complete in $\mathcal{F}^{k^*(co)}$ and $a \notin E$ holds, we deduce the existence of an argument c, s.t. $(c,a) \in R\left(\mathcal{F}^{k^*(co)}\right)$ and $(E,c) \notin R\left(\mathcal{F}^{k^*(co)}\right)$. Altogether, $(c,a) \in R\left(\mathcal{F}^{k^*(co)}\right)$ and $(E,c) \in R(\mathcal{F})\backslash R\left(\mathcal{F}^{k^*(co)}\right)$. Let d be the argument in E which attacks c, i.e. $(d,c) \in R(\mathcal{F})\backslash R\left(\mathcal{F}^{k^*(co)}\right)$. We observe that $(d,d) \in R(\mathcal{F})$ is impossible because $E \in cf(\mathcal{F})$ is assumed. Hence, $(c,c) \in R(\mathcal{F})$, $(c,d) \notin R(\mathcal{F})$ and $\{(a,a),(a,d),(d,a)\} \cap R(\mathcal{F}) \neq \varnothing$ follows. The cases $(a,a) \in R(\mathcal{F})$ and $(d,a) \in R(\mathcal{F})$ contradict the conflict-freeness of E in $\mathcal{F}$ because a is assumed to be defended by E in $\mathcal{F}$. In case of $(a,d) \in R(\mathcal{F})$ we use the already shown admissibility of E in $\mathcal{F}$ to infer $(E,a) \in R(\mathcal{F})$. Again, we get a contradiction to the conflict-freeness of E in $\mathcal{F}$ if we apply that a is defended by E in $\mathcal{F}$. □

The next lemma proves the robustness of the complete-*-kernel. That means, if two AFs $\mathcal{F}$ and $\mathcal{G}$ possess the same complete-*-kernel, then the same applies for any compositions $\mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \cup \mathcal{H}$ under the condition that the latter are strong expansions of there initial frameworks $\mathcal{F}$ and $\mathcal{G}$, respectively.

Lemma 8. *If $\mathcal{F}^{k^*(co)} = \mathcal{G}^{k^*(co)}$, then $(\mathcal{F} \cup \mathcal{H})^{k^*(co)} = (\mathcal{G} \cup \mathcal{H})^{k^*(co)}$ for all AFs $\mathcal{H}$ which satisfy $\mathcal{F} \preceq_S^N \mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \preceq_S^N \mathcal{G} \cup \mathcal{H}$.*

Proof. First notice that the assumption $\mathcal{F}^{k^*(co)} = \mathcal{G}^{k^*(co)}$ implies $A(\mathcal{F}) = A(\mathcal{F}^{k^*(co)}) = A(\mathcal{G}^{k^*(co)}) = A(\mathcal{G})$. Consider now an AF $\mathcal{H}$ satisfying the specified properties (strong expansion or equality). If $\mathcal{F} = \mathcal{F} \cup \mathcal{H}$, then $\mathcal{G} = \mathcal{G} \cup \mathcal{H}$ is implied (and vice versa). Consequently, in this case it is nothing to show because $(\mathcal{F} \cup \mathcal{H})^{k^*(co)} = (\mathcal{G} \cup \mathcal{H})^{k^*(co)}$ follows immediately. W.l.o.g. we may assume that $\mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \cup \mathcal{H}$ are indeed strong expansions of $\mathcal{F}$ or $\mathcal{G}$. Thus, $R(\mathcal{H}) \cap R(\mathcal{F}) = \varnothing$ and $R(\mathcal{H}) \cap R(\mathcal{G}) = \varnothing$ can be assumed. Let $(a,b) \in R\left((\mathcal{F} \cup \mathcal{H})^{k^*(co)}\right)$, therefore $(a,b) \in R(\mathcal{F} \cup \mathcal{H})$. We will show $(a,b) \in R\left((\mathcal{G} \cup \mathcal{H})^{k^*(co)}\right)$ by proof by cases (containedness of a and b in $A(\mathcal{F})$ or $A(\mathcal{H}) \backslash A(\mathcal{F})$). For all cases we will suppose $a \neq b$ since the self-loop case is obvi ous.

1st **case**: Let $a,b \in A(\mathcal{F})$. Assuming $(a,b) \in R\left(\mathcal{F}^{k^*(co)}\right)$ implies $(a,b) \in R\left(\mathcal{G}^{k^*(co)}\right)$ and therefore $(a,b) \in R(\mathcal{G})$. Consequently $(a,b) \in R\left((\mathcal{G} \cup \mathcal{H})^{k^*(co)}\right)$ holds since $\mathcal{G} \cup \mathcal{H}$ was assumed to be a strong expansion of $\mathcal{G}$ and so, no relevant attacks are added. The assumption $(a,b) \notin R\left(\mathcal{F}^{k^*(co)}\right)$ contradicts $(a,b) \in R\left((\mathcal{F} \cup \mathcal{H})^{k^*(co)}\right)$ because the reason to remove an attack remains untouched in $\mathcal{F} \cup \mathcal{H}$. **2**nd **case**: Let $a,b \in A(\mathcal{H}) \backslash A(\mathcal{F})$. Thus, $(a,b) \in R(\mathcal{G} \cup \mathcal{H})$ is implied. Suppose now $(a,b) \notin R\left((\mathcal{G} \cup \mathcal{H})^{k^*(co)}\right)$. This means, several reasons for removing have to be checked. The assumption $(a,a),(b,b) \in R(\mathcal{H})$ is inconsistent with $(a,b) \in R\left((\mathcal{F} \cup \mathcal{H})^{k^*(co)}\right)$. Thus, $(b,b) \in R(\mathcal{H})$, $(b,a) \notin R(\mathcal{H})$ and $\forall c \left((b,c) \in R(\mathcal{G} \cup \mathcal{H}) \rightarrow \{(a,c),(c,a),(c,c)\} \cap R(\mathcal{G} \cup \mathcal{H}) \neq \varnothing\right)$ has to hold. If there is no c in $A(\mathcal{G})$ which is attacked by b we deduce $(a,b) \notin R\left((\mathcal{F} \cup \mathcal{H})^{k^*(gr)}\right)$. Thus, consider an argument $c \in A(\mathcal{G})$ with the property $(b,c) \in R(\mathcal{H})$. Hence, $\{(a,c),(c,a),(c,c)\} \cap R(\mathcal{G} \cup \mathcal{H}) \neq \varnothing$ has to hold. In the first case, namely $(a,c) \in R(\mathcal{G} \cup \mathcal{H})$, $(a,c) \in R(\mathcal{H})$ and consequently $(a,c) \in R(\mathcal{F} \cup \mathcal{H})$ follows. The second case, i.e. $(c,a) \in R(\mathcal{G} \cup \mathcal{H})$, is just impossible since $\mathcal{G} \cup \mathcal{H}$ was assumed to be a strong expansion of $\mathcal{G}$. If $(c,c) \in R(\mathcal{G} \cup \mathcal{H})$, then $(c,c) \in R(\mathcal{G})$ and $(c,c) \in R(\mathcal{F})$ (since $\mathcal{F}^{k^*(co)} = \mathcal{G}^{k^*(co)}$ was assumed), therefore $(c,c) \in R(\mathcal{F} \cup \mathcal{H})$. In all cases we deduce $(a,b) \notin R\left((\mathcal{F} \cup \mathcal{H})^{k^*(co)}\right)$ contradicting the assumption. **3**rd **case**: Let $a \in A(\mathcal{H}) \backslash A(\mathcal{F})$ and $b \in A(\mathcal{F})$. Consequently, $(a,b) \in R(\mathcal{G} \cup \mathcal{H})$ holds. Assume now $(a,b) \notin R\left((\mathcal{G} \cup \mathcal{H})^{k^*(co)}\right)$. Again, several reasons for removing have to be considered. We observe that $(b,b) \in$

$R(\mathcal{G})$ (thus $(b,b) \in R(\mathcal{F})$) has to hold. If $(a,a) \in R(\mathcal{H})$ holds we deduce $(a,a),(b,b) \in R(\mathcal{F} \cup \mathcal{H})$ contrary to $(a,b) \in R\left((\mathcal{F} \cup \mathcal{H})^{k^*(co)}\right)$. Furthermore, we observe $(b,a) \notin R(\mathcal{F} \cup \mathcal{H}), R(\mathcal{G} \cup \mathcal{H})$ because $\mathcal{F} \cup \mathcal{H}$ and $\mathcal{G} \cup \mathcal{H}$ are assumed to be strong expansions of $\mathcal{F}$ and $\mathcal{G}$ respectively. Together with the assumption $(a,b) \in R\left((\mathcal{F} \cup \mathcal{H})^{k^*(co)}\right)$ we deduce the existence of an argument $c \in A(\mathcal{F})$, s.t. $(b,c) \in R(\mathcal{F}) \wedge \{(a,c),(c,a),(c,c)\} \cap R(\mathcal{F} \cup \mathcal{H}) = \emptyset$. Hence, $(c,c) \notin R(\mathcal{F})$ (therefore $(c,c) \notin R(\mathcal{G})$) and $(a,c),(c,a) \notin R(\mathcal{H})$ is implied. Consequently, if $(b,c) \in R(\mathcal{G})$, then $(a,b) \in R\left((\mathcal{G} \cup \mathcal{H})^{k^*(co)}\right)$ contradicting the assumption. Remember that $\mathcal{F}^{k^*(co)} = \mathcal{G}^{k^*(co)}$ has to hold. Hence, if $(b,c) \notin R(\mathcal{G})$, then (b,c) has to be deleted in $R\left(\mathcal{F}^{k^*(co)}\right)$. This is impossible because $(c,c) \notin R(\mathcal{F})$ is already shown. **4**th **case**: Let $a \in A(\mathcal{F})$ and $b \in A(\mathcal{H}) \backslash A(\mathcal{F})$. Here is nothing to show because $(a,b) \in R(\mathcal{F} \cup \mathcal{H})$ cannot hold if $\mathcal{F} \preceq_S^N \mathcal{F} \cup \mathcal{H}$ is fulfilled. □

Now we are prepared to show the main theorem for the case of complete semantics. The notion of complete-*-kernel is suitable to describe strong expansions equivalence w.r.t. complete semantics.

Theorem 8. *For any AFs $\mathcal{F}$, $\mathcal{G}$:*

$$\mathcal{F}^{k^*(co)} = \mathcal{G}^{k^*(co)} \Leftrightarrow \mathcal{F} \equiv^{co}_{\preceq_S^N} \mathcal{G}.$$

Proof. The first direction, namely $\mathcal{F}^{k^*(co)} = \mathcal{G}^{k^*(co)} \Rightarrow \mathcal{F} \equiv^{co}_{\preceq_S^N} \mathcal{G}$ can be shown by applying Lemmata 7 and 8 (similarly to Theorem 7). We will prove the only-if-direction by showing the contrapositive, i.e. $\mathcal{F}^{k^*(co)} \neq \mathcal{G}^{k^*(co)} \Rightarrow \mathcal{F} \not\equiv^{co}_{\preceq_S^N} \mathcal{G}$.

1st **case**: Suppose $A\left(\mathcal{F}^{k^*(co)}\right) \neq A\left(\mathcal{G}^{k^*(co)}\right)$. Thus, w.l.o.g. exists an argument $a \in A(\mathcal{F}) \backslash A(\mathcal{G})$. We define $\mathcal{H} = ((A(\mathcal{F}) \cup A(\mathcal{G})) \backslash \{a\}, \emptyset)$. Consider the existence of an extension $E \in \mathcal{E}_{co}(\mathcal{F} \cup \mathcal{H})$, s.t. $a \in E$ holds. Consequently, $E \notin \mathcal{E}_{co}(\mathcal{G} \cup \mathcal{H})$ and therefore $\mathcal{F} \not\equiv^{co}_{\preceq_S^N} \mathcal{G}$. Hence, we may assume that for all extensions $E \in \mathcal{E}_{co}(\mathcal{F} \cup \mathcal{H})$, $a \notin E$ holds. We define $\mathcal{H}' = \mathcal{H} \cup (\{a\}, \emptyset)$. Observe that a is unattacked in $\mathcal{H}$'. Since $\mathcal{G} \cup \mathcal{H}' = \mathcal{H}'$ we deduce that for any extension $E \in \mathcal{E}_{co}(\mathcal{G} \cup \mathcal{H}')$, $a \in E$ holds. Remember that the existence of a complete extension is guaranteed. Finally, since $\mathcal{E}_{co}(\mathcal{F} \cup \mathcal{H}) = \mathcal{E}_{co}(\mathcal{F} \cup \mathcal{H}')$ obviously holds we are done.

2nd **case**: Assume $R\left(\mathcal{F}^{k^*(co)}\right) \neq R\left(\mathcal{G}^{k^*(co)}\right)$ and $A\left(\mathcal{F}^{k^*(co)}\right) = A\left(\mathcal{G}^{k^*(co)}\right)$ ($= A(\mathcal{F}) = A(\mathcal{G})$). Thus, w.l.o.g. there exist some arguments $a, b \in A(\mathcal{F})$ with the property $(a,b) \in R\left(\mathcal{F}^{k^*(co)}\right) \backslash R\left(\mathcal{G}^{k^*(co)}\right)$. Let c be a fresh argument, i.e. $c \notin A(\mathcal{F})$. Furthermore we define

$$\mathcal{I} = \left(A(\mathcal{F}) \cup \{c\}, \{(c,c') \,|c' \in A(\mathcal{F}) \backslash \{a,b\}\}\right).$$

Case **2.1**: Let $a = b$ (self-loop case). Hence, $(a,a) \in R\left(\mathcal{F}^{k^*(co)}\right) \backslash R\left(\mathcal{G}^{k^*(co)}\right)$ and therefore $(a,a) \in R(\mathcal{F}) \backslash R(\mathcal{G})$ follows. We obtain $\{\{c\}\} = \mathcal{E}_{gr}(\mathcal{F} \cup \mathcal{I}) \neq$

$\mathcal{E}_{gr}(\mathcal{G} \cup \mathcal{I}) = \{\{a, c\}\}$. From now on we suppose that any self-loop is either contained in both $R\left(\mathcal{F}^{k^*(co)}\right)$ and $R\left(\mathcal{G}^{k^*(co)}\right)$ or in none of them.

Case **2.2**: Assume $a \neq b$. This means, $(a, b) \in R\left(\mathcal{F}^{k^*(co)}\right) \setminus R\left(\mathcal{G}^{k^*(co)}\right)$ and $(a, b) \in R(\mathcal{F})$. We will distinguish four cases for the presence or absence of the self-loops (a, a) and (b, b). Remember that $R(\mathcal{F}), R(\mathcal{G}), R\left(\mathcal{F}^{k^*(co)}\right)$ and $R\left(\mathcal{G}^{k^*(co)}\right)$ contain the same self-loops. Case **2.2.1**: $(a, a), (b, b) \in R(\mathcal{F})$. This case contradicts the assumption because $(a, b) \in R\left(\mathcal{F}^{k^*(co)}\right)$ cannot be fulfilled (compare complete-*-kernel). Case **2.2.2**: $(a, a), (b, b) \notin R(\mathcal{F})$. Observe that $(a, b) \notin R(\mathcal{G})$ holds because $(b, b) \notin R(\mathcal{G})$ and $(a, b) \notin R\left(\mathcal{G}^{k^*(co)}\right)$ was assumed. The attack (b, a) may or may not be an element of $R(\mathcal{F})$ or $R(\mathcal{G})$. In any case, $\{a, c\} \in \mathcal{E}_{co}(\mathcal{F} \cup \mathcal{I})$. This can be checked by applying splitting results. In the following we will leave this comment out. If $(b, a) \notin R(\mathcal{G})$, then $\{\{a, b, c\}\} = \mathcal{E}_{co}(\mathcal{G} \cup \mathcal{I})$ follows and if not, we deduce $\{\{b, c\}\} = \mathcal{E}_{co}(\mathcal{G} \cup \mathcal{I})$. Thus, $\mathcal{F} \not\equiv^{co}_{\preceq^N_S} \mathcal{G}$ is shown. Case **2.2.3**: $(a, a) \in R(\mathcal{F})$ and $(b, b) \notin R(\mathcal{F})$. Again, it is impossible that $(a, b) \in R(\mathcal{G})$ holds since $(b, b) \notin R(\mathcal{G})$ and $(a, b) \notin R\left(\mathcal{G}^{k^*(co)}\right)$ was assumed. The attack (b, a) may be contained in $R(\mathcal{F})$, $R(\mathcal{G})$ or not. In any case, $\{c\} \in \mathcal{E}_{co}(\mathcal{F} \cup \mathcal{I})$ and $\mathcal{E}_{co}(\mathcal{G} \cup \mathcal{I}) = \{\{b, c\}\}$ holds. Hence, $\mathcal{F} \not\equiv^{co}_{\preceq^N_S} \mathcal{G}$. Case **2.2.4**: $(a, a) \notin R(\mathcal{F})$ and $(b, b) \in R(\mathcal{F})$. Since $(a, b) \in R\left(\mathcal{F}^{k^*(gr)}\right)$ is assumed, we have to consider two sub-cases: First, $(b, a) \in R(\mathcal{F})$ and second, $(b, a) \notin R(\mathcal{F}) \wedge \exists c \in A(\mathcal{F}) : (b, c) \in R(\mathcal{F}) \wedge \{(a, c), (c, a), (c, c)\} \cap R(\mathcal{F}) = \varnothing$ (compare complete-*-kernel). If $(b, a) \in R(\mathcal{F})$, then $\{\{c\}, \{c, a\}\} = \mathcal{E}_{co}(\mathcal{F} \cup \mathcal{I})$ follows. Since $(a, b) \notin \mathcal{G}^{k^*(co)}$ is assumed, we deduce that if $(a, b) \in R(\mathcal{G})$, then $(b, a) \notin R(\mathcal{G})$ has to hold. In this case we obtain $\{\{c, a\}\} = \mathcal{E}_{co}(\mathcal{G} \cup \mathcal{I})$. Let $(a, b) \notin R(\mathcal{G})$. Hence, (b, a) may or may not be an element of $R(\mathcal{G})$. If (b, a) is contained in $R(\mathcal{G})$, $\{\{c\}\} = \mathcal{E}_{co}(\mathcal{G} \cup \mathcal{I})$ and if not, $\{\{a, c\}\} = \mathcal{E}_{co}(\mathcal{G} \cup \mathcal{I})$. Altogether, we have shown that in the first sub-case $\mathcal{F} \not\equiv^{co}_{\preceq^N_S} \mathcal{G}$ is implied. Consider now $(b, a) \notin R(\mathcal{F}) \wedge \exists c \in A(\mathcal{F}) : (b, c) \in R(\mathcal{F}) \wedge \{(a, c), (c, a), (c, c)\} \cap R(\mathcal{F}) = \varnothing$. Just like in case of grounded semantics, two possibilities for AF $\mathcal{F}$ remain. Again, we omit possible other arguments than a, b, and c.

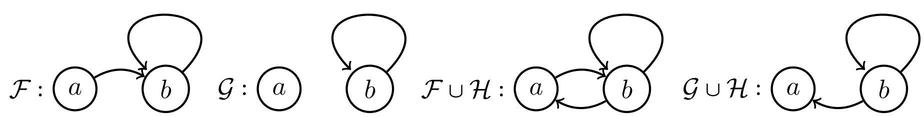
So far we know $(a, a), (c, c) \notin R(\mathcal{G})$ and $(b, b) \in R(\mathcal{G})$. Thus, there are $2^6 = 64$ combinations w.r.t. the presence and absence of $(a, b), (b, a), (b, c), (c, b), (a, c)$ and (c, a) in $R(\mathcal{G})$. Let d be a fresh argument. We define

$$\mathcal{I}' = (A(\mathcal{F}) \cup \{d\}, \{(d, c') \mid c' \in A(\mathcal{F}) \setminus \{a, b, c\}\}).$$

Observe that $\mathcal{E}_{co}(\mathcal{F}_1 \cup \mathcal{I}') = \mathcal{E}_{co}(\mathcal{F}_2 \cup \mathcal{I}') = \{\{a, c, d\}\}$ holds. If $(a, c) \in R(\mathcal{G})$ or $(c, a) \in R(\mathcal{G})$, then $\{a, c, d\} \notin \mathcal{E}_{co}(\mathcal{G} \cup \mathcal{I}')$ because complete extensions are conflict-free. Hence, we may assume $(a,c),(c,a) \notin R(\mathcal{G})$. This means, $2^4 = 16$ possibilities w.r.t. the presence or absence of $(a,b),(b,a),(b,c)$ and (c,b) remain. These sixteen AFs are listed in Theorem 6 (see pp. 21-22). $\mathcal{G}_6$ $(=\mathcal{F}_2)$ and $\mathcal{G}_8$ $(=\mathcal{F}_1)$ as well as $\mathcal{G}_{13}$, $\mathcal{G}_{14}$, $\mathcal{G}_{15}$ and $\mathcal{G}_{16}$ are impossible because $(a,b) \notin R\left(\mathcal{G}^{k^*(co)}\right)$ was assumed. Now, we notice by comparing to the proof of Theorem 8 that the remaining possibilities and $\mathcal{F}_1$ respectively $\mathcal{F}_2$ are not strong expansion equivalent w.r.t. the grounded semantics. Hence, by Prop. 2, item 2, $\mathcal{F} \not\equiv^{co}_{\preceq^N_S} \mathcal{G}$. □

We finish this section by giving an example showing that strong expansion equivalence and strong equivalence w.r.t. complete semantics do not coincide.

Example 11. On the one hand, the AFs $\mathcal{F}$ and $\mathcal{G}$ are strong expansion equivalent w.r.t. complete semantics because $\mathcal{F}^{k^*(co)} = \mathcal{G}^{k^*(co)} = \mathcal{G}$ (Theorem 8). On the other hand, they are not equivalent w.r.t. arbitrary expansions which can be made explicit by conjoining them with $\mathcal{H} = (\{a,b\},\{(b,a)\})$. We have $\{a\} \in \mathcal{E}_{co}(\mathcal{F} \cup \mathcal{H})$ and $\{a\} \notin \mathcal{E}_{co}(\mathcal{G} \cup \mathcal{H})$.



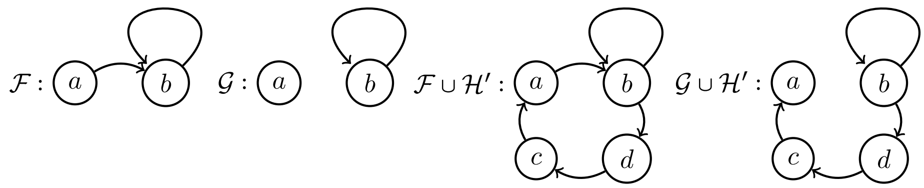
5. Characterizing Normal Expansion Equivalence

The main aim of this section is the characterization of normal expansion equivalence w.r.t. stable, semi-stable, admissible, preferred, ideal, grounded and complete semantics. We will see that there is no need for further novel kernel definitions. In fact, the main and quite surprisingly result for the considered semantics can be briefly and concisely presented in the following "equality":

$$\text{normal expansion equivalence} = \text{strong equivalence}$$

This means, if two AFs $\mathcal{F}$ and $\mathcal{G}$ are proven to be normal expansion equivalent, then the requirement that $\mathcal{F}$ and $\mathcal{G}$ are equivalent when conjoined with any further framework $\mathcal{H}$ is fulfilled too. This is quite surprising since the class of normal expansions is obviously a proper subset of the class of arbitrary expansions. In other words, if different implicit information of two AFs $\mathcal{F}$ and $\mathcal{G}$ is made explicit by conjoining them with an AF $\mathcal{H}$ which adds further attacks between former arguments, then there exists an AF $\mathcal{H}$' showing this difference without changing the former attack-relations of $\mathcal{F}$ and $\mathcal{G}$. Consider the following example.

Example 12 (Example 11 cont.). In Example 11 we observed that $\mathcal{F}$ and $\mathcal{G}$ are not strongly equivalent w.r.t. complete semantics. This property can be shown by conjoining them with $\mathcal{H}' = (\{a,b,c,d\},\{(b,d),(c,a),(d,c)\})$ which do not add further attacks between the old arguments a and b. Note that $\{a,d\} \in \mathcal{E}_{co}(\mathcal{F} \cup \mathcal{H}')$ and $\{a,d\} \notin \mathcal{E}_{co}(\mathcal{G} \cup \mathcal{H}')$.



5.1. Normal Expansion Equivalence for Stable Semantics

At first we consider stable semantics. Remember that we have already shown (Section 4.1) that the notion of *st*-kernel not only characterizes strong equivalence but also strong expansion equivalence w.r.t. stable semantics. In consideration that the class of normal expansions lie inbetween (w.r.t. subset-relation) the classes of arbitrary and strong expansions the following theorems follow immediately.

Theorem 9. *For any AFs $\mathcal{F}$, $\mathcal{G}$,*

$$\mathcal{F}^{k(st)} = \mathcal{G}^{k(st)} \Leftrightarrow \mathcal{F} \equiv^{st}_{\preceq^N} \mathcal{G}.$$

Proof. Combine Theorems 1, 4 and Prop. 3.

5.2. Normal Expansion Equivalence for Semi-Stable, Admissible, Preferred and Ideal Semantics

Oikarinen and Woltran showed that the *ad*-kernel serves as a uniform characterization for strong equivalence w.r.t. semi-stable, admissible, preferred and ideal semantics (Theorems 1, 3). The following theorem proves that this result carries over to normal expansion equivalence. Remember that the *ad*-kernel of an AF $\mathcal{F} = (A,R)$ is $\mathcal{F}^{k(ad)} = (A, R\backslash\{(a,b) \mid a \neq b, (a,a) \in R, \{(b,a),(b,b) \cap R \neq \varnothing\}\})$.

Theorem 10. *For any AFs $\mathcal{F}$, $\mathcal{G}$ and $\sigma \in \{ss, ad, pr, id\}$,*

$$\mathcal{F}^{k(ad)} = \mathcal{G}^{k(ad)} \Leftrightarrow \mathcal{F} \equiv^{\sigma}_{\preceq^N} \mathcal{G}.$$

Proof. In case of semi-stable semantics the assertion follows by combining Theorems 1, 3, 5 and Prop. 3.

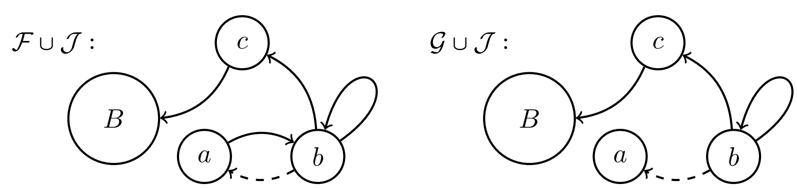
Let us turn to admissible, preferred and ideal semantics ($\sigma \in \{ad, pr, id\}$). Note that Theorems 1, 3 and Prop. 3 imply $\mathcal{F}^{k(ad)} = \mathcal{G}^{k(ad)} \Rightarrow \mathcal{F} \equiv^{\sigma}_{\preceq^N_S} \mathcal{G}$. This means, it suffices to show that $\mathcal{F}^{k(ad)} \neq \mathcal{G}^{k(ad)} \Rightarrow \mathcal{F} \not\equiv^{\sigma}_{\preceq^N_S} \mathcal{G}$. Almost all cases are already proven in Theorem 5 (compare footnote 5 on page 13). We will prove now the remaining case 2.2.4.

The case **2.2.4** is based on the following assumptions: $A(\mathcal{F}^{k(ad)}) = A(\mathcal{G}^{k(ad)})$ and $R(\mathcal{F}^{k(ad)}) \neq R(\mathcal{G}^{k(ad)})$. Hence, there are two arguments $a, b \in A(\mathcal{F})$, s.t.

$(a,b) \in R(\mathcal{F}^{k(ad)}) \setminus R(\mathcal{G}^{k(ad)})$. Furthermore, we assume $a \neq b$ and b is self-defeating, i.e. $(a,a) \notin R(\mathcal{F})$ and $(b,b) \in R(\mathcal{F})$. Using the following AF $\mathcal{J}$ we will prove that the AFs $\mathcal{F}$ and $\mathcal{G}$ are not normal expansion equivalent w.r.t. admissible, preferred and ideal semantics. Let c be a fresh argument and $B = A(\mathcal{F}) \setminus \{a,b\}$, then

$$\mathcal{J} = (A(\mathcal{F}) \cup \{c\}, \{(c,c') \mid c' \in B\} \cup \{(b,c)\}).$$

The following figure illustrates $\mathcal{F} \cup \mathcal{J}$ and $\mathcal{G} \cup \mathcal{J}$. Note that (b,a) may or not be in $R(\mathcal{F})$ or $R(\mathcal{G})$ (indicated by dashed arrows). Furthermore, $(a,b) \notin R(\mathcal{G})$ since $(a,a) \notin R(\mathcal{G})$ and $(a,b) \notin R(\mathcal{G}^{k(ad)})$ was assumed. For reasons of clarity we left out possible attacks between the arguments in B and $\{a,b\}$.



Whether (b,a) is in $R(\mathcal{F})$ or not we obtain $\mathcal{E}_{pr}(\mathcal{F} \cup \mathcal{J}) = \{\{a,c\}\}$. If $(b,a) \in R(\mathcal{G})$ we observe that apart from the empty set no other set is admissible in $\mathcal{G} \cup \mathcal{J}$. Hence, $\mathcal{E}_{pr}(\mathcal{G} \cup \mathcal{J}) = \{\emptyset\}$. Consider now $(b,a) \notin R(\mathcal{G})$. The preferred extension of $\mathcal{G} \cup \mathcal{J}$ depends on whether a defends itself in $\mathcal{G}$ (and so in $\mathcal{G} \cup \mathcal{J}$) or not. If so, we have $\mathcal{E}_{pr}(\mathcal{G} \cup \mathcal{J}) = \{\{a\}\}$. If not, we get $\mathcal{E}_{pr}(\mathcal{G} \cup \mathcal{J}) = \{\emptyset\}$. In all cases the preferred extension of $\mathcal{G} \cup \mathcal{J}$ is unique and differs from $\{a,c\}$. Thus, the same holds for ideal and admissible semantics (Prop. 1, 2). Hence, $\mathcal{F} \not\equiv^{\sigma}_{\preceq^N} \mathcal{G}$ for $\sigma \in \{ad, pr, id\}$ concluding the proof. □

5.3. Normal Expansion Equivalence for Grounded Semantics

Strong equivalence w.r.t. the very cautious grounded semantics can be captured by the gr-kernel which "identifies" attacks (a,b) as redundant if both a and b are self-attacking or there is an attack (b,a) and b is self-defeating (Theorem 1). The latter condition is a unique feature of the grounded semantics and reflects its subset-minimality among the complete extensions. The following theorem shows that the gr-kernel is even suitable to characterize normal expansion equivalence w.r.t. grounded semantics. For convenience, we repeat the definition of the gr-kernel of an AF $\mathcal{F} = (A,R)$, namely $\mathcal{F}^{k(gr)} = (A, R \setminus \{(a,b) \mid a \neq b, (b,b) \in R, \{(a,a),(b,a) \cap R \neq \emptyset\}\})$.

Theorem 11. *For any AFs $\mathcal{F}$ and $\mathcal{G}$,*

$$\mathcal{F}^{k(gr)} = \mathcal{G}^{k(gr)} \Leftrightarrow \mathcal{F} \equiv^{gr}_{\preceq^N} \mathcal{G}.$$

Proof. The if-direction, namely $\mathcal{F}^{k(gr)} = \mathcal{G}^{k(gr)} \Rightarrow \mathcal{F} \equiv^{gr}_{\preceq^N} \mathcal{G}$ is a consequence of Theorem 2 and Prop. 3. Hence, it suffices to show the only-if-direction, i.e. $\mathcal{F} \equiv^{gr}_{\preceq^N} \mathcal{G} \Rightarrow \mathcal{F}^{k(gr)} = \mathcal{G}^{k(gr)}$. We will prove the contrapositive.

Suppose $\mathcal{F}^{k(gr)} \neq \mathcal{G}^{k(gr)}$. We skip the consideration of different arguments, i.e. $\mathbf{1}^{st}$ **case**: $A(\mathcal{F}^{k(gr)}) \neq A(\mathcal{G}^{k(gr)})$ as well as the occurrence of different self-loops, i.e. case **2.1**: $(a,b) \in R(\mathcal{F}^{k(gr)})\backslash R(\mathcal{G}^{k(gr)})$ where $a = b$ holds since the proofs of them are exactly the same as in Theorem 7. In the following we assume $A(\mathcal{F}^{k(gr)}) = A(\mathcal{G}^{k(gr)})$ and any self-loop is either contained in both $R(\mathcal{F}^{k(gr)})$ and $R(\mathcal{G}^{k(gr)})$ or none of them.

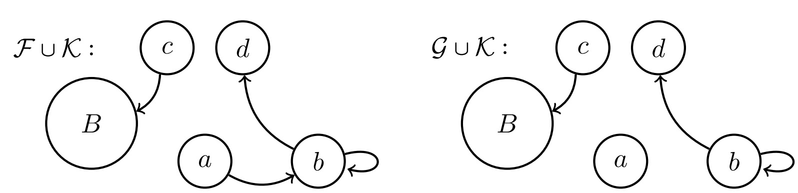
Case **2.2**: Consider $(a,b) \in R(\mathcal{F}^{k(gr)})\backslash R(\mathcal{G}^{k(gr)})$ with $a \neq b$. Note that $(a,b) \in R(\mathcal{F})$ is implied. Consequently, at least one of the following two statements has to hold: $(b,b) \notin R(\mathcal{F})$; $(a,a) \notin R(\mathcal{F})$ and $(b,a) \notin R(\mathcal{F})$. We will use the standard construction $\mathcal{I}$ to prove some cases (c is a fresh argument).

$$\mathcal{I} = (A(\mathcal{F}) \cup \{c\}, \{(c,c') \mid c' \in A(\mathcal{F})\backslash\{a,b\}\}) .$$

Case **2.2.1**: Let $(b,b) \notin R(\mathcal{F})$. Consequently, $(b,b) \notin R(\mathcal{G})$ and hence, $(a,b) \notin R(\mathcal{G})$ since $(a,b) \notin R(\mathcal{G}^{k(gr)})$ holds. The following extensions can be obtained by applying splitting results. If $(b,a) \in R(\mathcal{F})$ or $(a,a) \in R(\mathcal{F})$, then $\mathcal{E}_{gr}(\mathcal{F} \cup \mathcal{I}) = \{\{c\}\}$ holds. If not, we get $\mathcal{E}_{gr}(\mathcal{F} \cup \mathcal{I}) = \{\{a,c\}\}$. On the other hand, if $(b,a) \in R(\mathcal{G})$ or $(a,a) \in R(\mathcal{G})$ holds, we obtain $\mathcal{E}_{gr}(\mathcal{G} \cup \mathcal{I}) = \{\{b,c\}\}$. If not, we conclude $\mathcal{E}_{gr}(\mathcal{G} \cup \mathcal{I}) = \{\{a,b,c\}\}$. This means, for all possible combinations $\mathcal{E}_{gr}(\mathcal{F} \cup \mathcal{I}) \neq \mathcal{E}_{gr}(\mathcal{G} \cup \mathcal{I})$ follows. Case **2.2.2**: Let $(b,b) \in R(\mathcal{F})$ and furthermore, $(a,a) \notin R(\mathcal{F})$ and $(b,a) \notin R(\mathcal{F})$. Note that the attacks in $\mathcal{F}$ w.r.t. the arguments a and b are uniquely determined. The union of $\mathcal{F}$ and $\mathcal{I}$ yields $\{a,c\}$ as the unique grounded extension. The AF $\mathcal{G}$ may occur in three configurations (remember that $(a,a) \notin R(\mathcal{G})$ and $(b,b) \in R(\mathcal{G})$ is already assumed), namely i) $(a,b),(b,a) \in R(\mathcal{G})$, ii) $(a,b) \notin R(\mathcal{G})$, $(b,a) \in R(\mathcal{G})$ and iii) $(a,b),(b,a) \notin R(\mathcal{G})$. In the first two cases $\mathcal{E}_{gr}(\mathcal{G} \cup \mathcal{H}) = \{\{c\}\}$ is implied. The third possibility establishes the grounded extension $\{a,c\}$ too. Hence, we have to find another AF $\mathcal{K}$, s.t. $\mathcal{E}_{gr}(\mathcal{F} \cup \mathcal{K}) \neq \mathcal{E}_{gr}(\mathcal{G} \cup \mathcal{K})$ is implied. Let c and d be fresh arguments and $B = A(\mathcal{F})\backslash\{a,b\}$. We define

$$\mathcal{K} = (A(\mathcal{F}) \cup \{c,d\}, \{(b,d)\} \cup \{(c,c') \mid c' \in B\}) .$$

The following figure illustrates $\mathcal{F} \cup \mathcal{K}$ and $\mathcal{G} \cup \mathcal{K}$. We left out possible attacks between the arguments in B and $\{a,b\}$ since they can be "ignored" in case of evaluating the AFs w.r.t. grounded semantics .



The grounded extensions of $\mathcal{F} \cup \mathcal{K}$ and $\mathcal{G} \cup \mathcal{K}$ differ, namely $\{\{a,c,d\}\} = \mathcal{E}_{gr}(\mathcal{F} \cup \mathcal{K}) \neq \mathcal{E}_{gr}(\mathcal{G} \cup \mathcal{K}) = \{\{a,c\}\}$. This can be seen as follows: The argument c is unattacked in $\mathcal{F} \cup \mathcal{K}$ and $\mathcal{G} \cup \mathcal{K}$. Furthermore, in both AFs a is defended by $\{c\}$ and hence, $\{a,c\}$ has to be a subset of the grounded extension in both AFs. We observe that d has to belong to the grounded extension of $\mathcal{F}\cup\mathcal{K}$ since it is defended by $\{a,c\}$. This does not apply to $\mathcal{G}\cup\mathcal{K}$. Consequently, $\mathcal{F} \not\equiv^{gr}_{\preceq^N} \mathcal{G}$ concluding the proof. □

5.4. Normal Expansion Equivalence for Complete Semantics

Finally, we turn to complete semantics. Similarly to the other semantics considered in this paper we will show that a different semantical behaviour of two AFs w.r.t. arbitrary expansions and complete semantics is sufficient for not being normal expansion equivalent w.r.t. complete semantics. This claim is illustrated in Example 12 from the beginning of this section and will be a consequence of the following theorem showing that the *co*-kernel adequately describes normal expansion equivalence w.r.t. complete semantics. Remember that the *co*-kernel of an AF $\mathcal{F}=(A,R)$ do not possess an attack $(a,b)\in R$ if and only if both a and b are self-attacking.

Theorem 12. *For any AFs $\mathcal{F}$ and $\mathcal{G}$,*

$$\mathcal{F}^{k(co)} = \mathcal{G}^{k(co)} \Leftrightarrow \mathcal{F} \equiv^{co}_{\preceq^N} \mathcal{G}.$$

Proof. The if-direction, namely $\mathcal{F}^{k(co)} = \mathcal{G}^{k(co)} \Rightarrow \mathcal{F} \equiv^{co}_{\preceq^N} \mathcal{G}$ can be obtained by combining Theorem 1 and Prop. 3, item 1. Hence, it suffices to show that $\mathcal{F} \equiv^{co}_{\preceq^N} \mathcal{G} \Rightarrow \mathcal{F}^{k(co)} = \mathcal{G}^{k(co)}$ holds. By Theorem 2 the latter implication is equivalent to $\mathcal{F} \equiv^{co}_{\preceq^N} \mathcal{G} \Rightarrow \mathcal{F}^{k(ad)} = \mathcal{G}^{k(ad)}$ and $\mathcal{F}^{k(gr)} = \mathcal{G}^{k(gr)}$. Using Theorems 10, 11 we may replace the kernel-equalities by normal expansion equivalence w.r.t. preferred or complete semantics. Thus, we obtain the implication we have to prove, $\mathcal{F} \equiv^{co}_{\preceq^N} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{pr}_{\preceq^N} \mathcal{G}$ and $\mathcal{F} \equiv^{gr}_{\preceq^N} \mathcal{G}$. If $\mathcal{F} \not\equiv^{pr}_{\preceq^N} \mathcal{G}$ (or $\mathcal{F} \not\equiv^{gr}_{\preceq^N} \mathcal{G}$), then there exists an AF $\mathcal{H}$ with the property $\mathcal{F} \preceq^N \mathcal{F}\cup\mathcal{H}$ and $\mathcal{G} \preceq^N \mathcal{G}\cup\mathcal{H}$, s.t. $\mathcal{F}\cup\mathcal{H} \not\equiv^{pr} \mathcal{G}\cup\mathcal{H}$ (or $\mathcal{F}\cup\mathcal{H} \not\equiv^{gr} \mathcal{G}\cup\mathcal{H}$). Hence, in both cases $\mathcal{F}\cup\mathcal{H} \not\equiv^{co} \mathcal{G}\cup\mathcal{H}$ by applying the contrapositive of Prop. 2, item 2. Consequently, $\mathcal{F} \not\equiv^{co}_{\preceq^N} \mathcal{G}$ is shown concluding the proof. □

6. Summary of Results and Further Conclusions

6.1. Overview: "Strength" of Kernels

In this subsection we want to provide a quick overview of the considered kernels as well as their potential w.r.t. characterizing equivalence relations. In the following we recall the resulting attack-relation of the σ-kernel or σ-*-kernel of an AF $\mathcal{F}=(A,R)$. Remember that the considered kernels do not change the initial set of arguments, i.e. $\mathcal{F}^{k(\sigma)} = (A, R^{k(\sigma)})$ or $\mathcal{F}^{k^*(\sigma)} = (A, R^{k^*(\sigma)})$, respectively.

1. $R^{k(st)} = R \setminus \{(a,b) \mid a \neq b, (a,a) \in R\}$,
2. $R^{k(ad)} = R \setminus \{(a,b) \mid a \neq b, (a,a) \in R, \{(b,a),(b,b)\} \cap R \neq \varnothing\}$,

3. $R^{k^*(ad)} = R \setminus \{(a,b) \mid a \neq b, ((a,a) \in R \wedge \{(b,a),(b,b)\} \cap R \neq \varnothing) \vee$
$((b,b) \in R \wedge \forall c\ ((b,c) \in R \rightarrow \{(a,c),(c,a),(c,c),(c,b)\} \cap R \neq \varnothing))\}$,
4. $R^{k(gr)} = R \setminus \{(a,b) \mid a \neq b, (b,b) \in R, \{(a,a),(b,a)\} \cap R \neq \varnothing\}$,
5. $R^{k^*(gr)} = R \setminus \{(a,b) \mid a \neq b, ((b,b) \in R \wedge \{(a,a),(b,a)\} \cap R \neq \varnothing) \vee$
$((b,b) \in R \wedge \forall c\ ((b,c) \in R \rightarrow \{(a,c),(c,a),(c,c)\} \cap R \neq \varnothing))\}$,
6. $R^{k(co)} = R \setminus \{(a,b) \mid a \neq b, (a,a),(b,b) \in R\}$,
7. $R^{k^*(co)} = R \setminus \{(a,b) \mid a \neq b, ((a,a),(b,b) \in R) \vee$
$((b,b) \in R \wedge (b,a) \notin R \wedge \forall c\ ((b,c) \in R \rightarrow \{(a,c),(c,a),(c,c)\} \cap R \neq \varnothing))\}$.

The following table provides a comprehensive overview of the potential of the above mentioned kernels. For the sake of completeness we also mentioned local expansion equivalence (second line) since Oikarinen and Woltran have shown that the *ad*-kernel even characterizes local expansion equivalence w.r.t. semi-stable, admissible, preferred and ideal semantics (Theorem 8 in [7]). The entry "[7],n" in the table indicates two facts: First, the characterization problem is already solved in [7] (Theorem n) and second, none of the considered kernels serve as a characterization.

	st	ss	ad	pr	id	gr	co
$\mathcal{F} \equiv^{\sigma}_{\leq L} \mathcal{G}$	$[7], 9$	$k(ad)$	$k(ad)$	$k(ad)$	$k(ad)$	$[7], 10$	$[7], 11$
$\mathcal{F} \equiv^{\sigma}_{\leq} \mathcal{G}$	$k(st)$	$k(ad)$	$k(ad)$	$k(ad)$	$k(ad)$	$k(gr)$	$k(co)$
$\mathcal{F} \equiv^{\sigma}_{\leq N} \mathcal{G}$	$k(st)$	$k(ad)$	$k(ad)$	$k(ad)$	$k(ad)$	$k(gr)$	$k(co)$
$\mathcal{F} \equiv^{\sigma}_{\leq^N_S} \mathcal{G}$	$k(st)$	$k(ad)$	$k^*(ad)$	$k^*(ad)$	$k^*(ad)$	$k^*(gr)$	$k^*(co)$

Figure 2: "Strength" of Kernels

6.2. The Role of Self-loop-free AFs

If we take a closer look at the definitions of the σ-kernel or σ-*-kernel of an AF $\mathcal{F}$ we observe that in case of self-loop-free AFs nothing changes, i.e. $\mathcal{F}$ and its corresponding kernel are identical. Consequently, any equivalence relation on the set of AFs characterizable through equality of kernels presented in this paper collapses to identity if we restrict ourselves to self-loop-free AFs. This is stated in the following proposition for the equivalence relation studied in this paper, namely normal and strong expansion equivalence.

Proposition 4. *For any self-loop-free AFs $\mathcal{F}$, $\mathcal{G}$, any $\sigma \in \{st, ss, ad, pr, id, gr, co\}$ and any $\Phi \in \{\preceq^N, \preceq^N_S\}$:*

$$\mathcal{F} = \mathcal{G} \text{ iff } \mathcal{F} \equiv^\sigma_\Phi \mathcal{G}.$$

This means, self-loop-free AFs are redundancy-free or in other words, all attacks may play a crucial role w.r.t. further evaluations provided that the expansions are normal or strong. In the introductory part of this paper we noted that such kinds of expansions naturally occur if Dung-style AFs are (re-)instantiated by a deductive argumentation system where a new piece of information was added to the knowledge base. We want to mention that there are some formalisms like classical logic-based frameworks where self-attacking arguments do not occur (cf. Theorem 4.13 in [4]). Other argumentation systems like ASPIC [28] or a very simple formalism presented by Caminada in [29] "allow" self-defeating arguments. In the latter systems arguments are defined by applying two kinds of inference rules, namely strict and defeasible rules. We refer the reader to Section 7 in [28] or Section 3 in [29] for examples of self-defeating arguments.

As an aside, the result stated in Prop. 4 cannot be conveyed to local and weak expansion equivalence w.r.t. any considered semantics. This means, there are syntactically different and self-loop-free AFs which are local expansion equivalent (cf. Example 16 in [7]) or weak expansion equivalent (cf. Example 4 in [12]).

6.3. Relations Between Different Notions of Equivalence

In Subsection 3.4 (Prop. 3, Figure 1) we considered preliminary relations between several notions of equivalence which hold for **any** semantics. Using the characterization theorems proved in [7, 12] as well as the established results of this paper we may provide a more fine-grained picture for the considered semantics.

We will present the results in one single theorem. For a better understanding we provide arrowed diagrams just like in Figure 1. The obtained relations hardly need a proof since they are simply combinations of former theorems. For this reason we only list the involved statements instead of providing full proofs.

Theorem 13. *For any AFs $\mathcal{F}$ and $\mathcal{G}$,*

1. $\mathcal{F} \equiv^{st}_{\preceq} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv^{st}_{\preceq^N} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv^{st}_{\preceq^N_S} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{st}_{\preceq^L} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{st}_{\preceq^N_W} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{st} \mathcal{G}$
2. $\mathcal{F} \equiv^{ss}_{\preceq} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv^{ss}_{\preceq^N} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv^{ss}_{\preceq^N_S} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv^{ss}_{\preceq^L} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{st}_{\preceq^N_W} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{st} \mathcal{G}$
3. $\mathcal{F} \equiv^{ad}_{\preceq} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv^{ad}_{\preceq^N} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv^{ad}_{\preceq^L} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{ad}_{\preceq^N_S} \mathcal{G}, \mathcal{F} \equiv^{ad}_{\preceq^N_W} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{ad} \mathcal{G}$
4. $\mathcal{F} \equiv^{pr}_{\preceq} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv^{pr}_{\preceq^N} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv^{pr}_{\preceq^L} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{pr}_{\preceq^N_S} \mathcal{G}, \mathcal{F} \equiv^{pr}_{\preceq^N_W} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{pr} \mathcal{G}$
5. $\mathcal{F} \equiv^{id}_{\preceq} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv^{id}_{\preceq^N} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv^{id}_{\preceq^L} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{id}_{\preceq^N_S} \mathcal{G}, \mathcal{F} \equiv^{id}_{\preceq^N_W} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{id} \mathcal{G}$
6. $\mathcal{F} \equiv^{gr}_{\preceq} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv^{gr}_{\preceq^N} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{gr}_{\preceq^L} \mathcal{G}, \mathcal{F} \equiv^{gr}_{\preceq^N_S} \mathcal{G}, \mathcal{F} \equiv^{gr}_{\preceq^N_W} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{gr} \mathcal{G}$

7. $\mathcal{F} \equiv^{co}_{\preceq} \mathcal{G} \Leftrightarrow \mathcal{F} \equiv^{co}_{\preceq^N} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{co}_{\preceq^L} \mathcal{G}, \mathcal{F} \equiv^{co}_{\preceq^N_S} \mathcal{G}, \mathcal{F} \equiv^{co}_{\preceq^N_W} \mathcal{G} \Rightarrow \mathcal{F} \equiv^{co} \mathcal{G}$

Proof. We only list the involved statements.
ad 1.) Combine Prop. 3, Theorems 1, 4, 9, Theorem 9 in [7] and Prop. 3 in [12]
ad 2.) Combine Prop. 3, Theorems 1, 3, 5, 10 and Theorem 8 in [7]
ad 3.-5.) Combine Prop. 3, Theorems 1, 3, 6, 10 and Theorem 8 in [7]
ad 6.) Combine Prop. 3, Theorems 1, 3, 7, 11
ad 7.) Combine Prop. 3, Theorems 1, 3, 8, 12 □

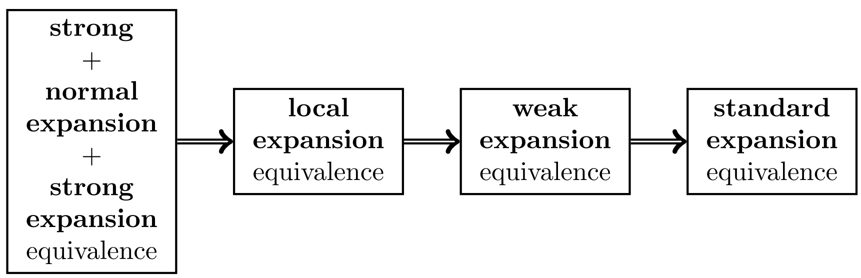


Figure 3: Relations for Stable Semantics

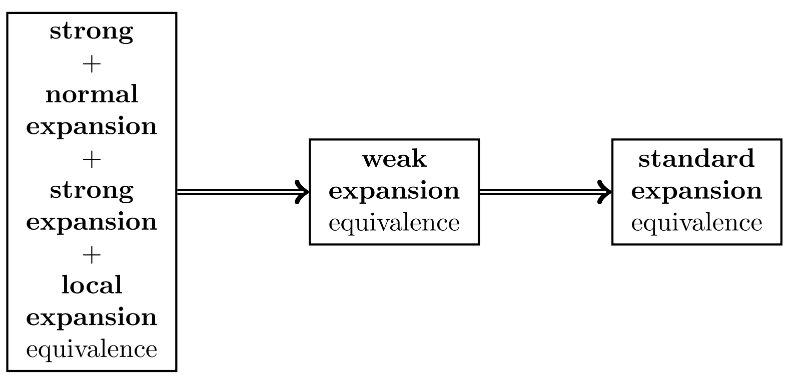


Figure 4: Relations for Semi-stable Semantics

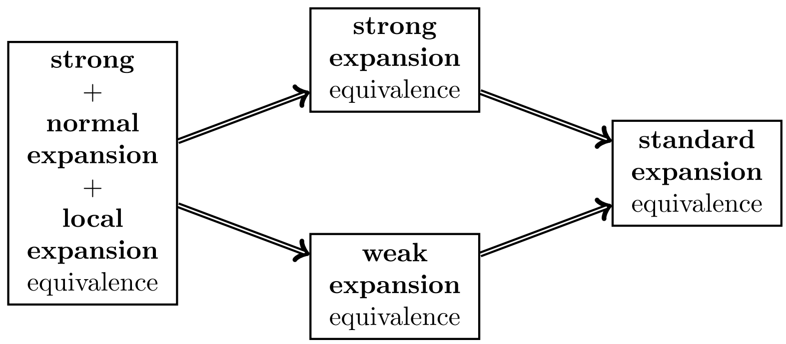


Figure 5: Relations for Admissible, Preferred and Ideal Semantics

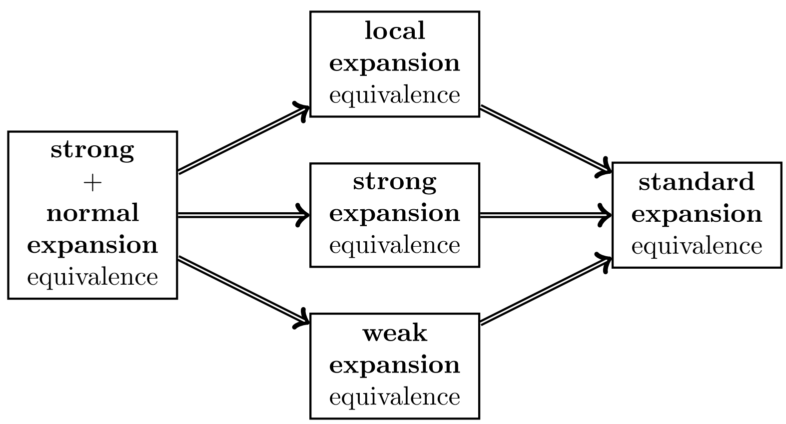


Figure 6: Relations for Grounded and Complete Semantics

In our point of view, the most remarkable relations are those of stable and semi-stable semantics since their corresponding equivalence relations are totally ordered w.r.t. subset-relation. Bearing in mind that strong, weak and local expansions are completely different concepts the containedness or coincidence of their corresponding equivalence relations is unexpected. We would like to point out that there are as yet no characterization theorems for weak expansion equivalence w.r.t. admissible, preferred, ideal, grounded and complete semantics. This means, Figure 5 and Figure 6 are not necessarly final pictures.

Baroni and Giacomin [30] introduced several general criteria for comparing and evaluating semantics. This paper was an important step to classify semantics because until its publication comparisons between semantics were almost exclusively example driven. The figures above motivate further criteria to compare argumentation semantics on an abstract level, for example, coincidence, containedness or incomparability (w.r.t. subset-relation) of equivalence relations. The study of such *equivalence-based criteria* as well as their relations to the criteria proposed in [30] will be part of future work.

7. Discussion and Related Work

In this paper, we studied two new equivalence relations for AFs, namely strong and normal expansion equivalence which lie in-between standard equivalence and the recently proposed strong equivalence [7]. We provided characterization theorems for a representative set of admissible-based semantics, namely stable, semi-stable, admissible, preferred, ideal, complete and grounded semantics. In particulur, we showed that for any considered semantics, normal expansion equivalence coincides with strong equivalence. It is part of future work to verify whether this observation conveys to semantics based on conflict-free sets like stage and CF2 semantics [31, 32]. Furthermore, we showed that strong expansion equivalence is strictly coarser than normal expansion equivalence in case of admissible, preferred, ideal, complete and grounded semantics, while stable as well as semi-stable semantics do not "distinguish" between strong and normal expansion equivalence.

The obtained characterization theorems are based on syntactical criteria. To determine wheter two AFs are normal or strong expansion equivalent w.r.t. a certain semantics σ it suffices to compare certain *kernels* of them. A kernel of an AF $\mathcal{F}$ is itself an AF obtained from $\mathcal{F}$ by deleting certain attacks depending on the considered semantics σ. It has been shown that, if two AFs possess identical kernels under a certain semantics σ then they are inter-substitutable w.r.t. further evaluations in dynamic scenarios satisfying the concept of normal or strong expansions, respectively. Such replacement properties are essential for logical approaches in general, particularly for non-monotonic logics where this question becomes a non-trivial task. In fact, beside the additional theoretical insights into how Dung's non-monotonic formalism works, the results may be used to refine existing algorithms to compute extensions. A good example for simplifying AFs or reducing complexity in terms of number of attacks is Example 8 on page 24. Although the AF $\mathcal{F}$ possesses much more attacks than AF $\mathcal{G}$ they are strong expansion (and thus also standard) equivalent w.r.t. admissible, preferred and ideal semantics. It should be noted that during such a simplification process no arguments can be deleted. Furthermore, in case of self-loop-free AFs normal and strong expansion equivalence collapse to identity, i.e. every attack may play a crucial role w.r.t. further evaluations.

In contrast to other non-monotonic formalisms where a huge number of equivalence notions inbetween standard and strong equivalence were studied, e.g. query equivalence [33] and uniform equivalence [34] in case of logic programs, we are not aware of further studies apart from [7, 14] devoted to abstract argumentation. In [35] different notions of equivalence w.r.t. stable semantics of two logic-based argumentation systems is studied. More precisely, they studied the question when two systems, not necessarily built over the same knowledge base and/or not necessarily use the same attack definition, produce the same output w.r.t. stable semantics. Our results as well as the characterization theorems in [7, 14] are in some sense useless for their aim. The reason is that the authors concentrated on classical logic-based argumentation systems where self-attacking arguments provably do not occur (cf. Theorem 4.13 in [4]). As mentioned in Section 6.2 other formalisms like ASPIC [28] “allow” self-defeating arguments. Consequently, identifying redundant attacks may simplify the evaluation of such systems.

Another mentionable work dealing with various notions of equivalence with regard to deductive argumentation is [36]. Due to the use of a very basic definition of an argument the presented complexity results holds for a whole range of argumentation systems. They showed, for instance, that checking equivalence of argument sets is not computationally harder than checking equivalence of arguments. Both are co-NP-complete.

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